102. *On Compactness of Weak Topologies*

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Let R be a space and $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$ a system of mappings of R into topological spaces S_λ with neighbourhood systems $\mathfrak{N}_\lambda$ $(\lambda \in \Lambda)$. Concerning the weak topology of R by $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$, i.e. the weakest topology of R for which all $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$ are continuous, we have (H. Nakano: Topology and Linear Topological Spaces, Tokyo (1951), §19, Theorem 4. This book will be denoted by TLTS):

Theorem 1. *If all S_λ $(\lambda \in \Lambda)$ are compact Hausdorff spaces, then, in order that the weak topology of R be compact, it is necessary and sufficient that for any system of points $a_\lambda \in S_\lambda$ $(\lambda \in \Lambda)$ subject to the condition*

$$\text{(F)} \qquad \bigcap_{\nu=1}^{n} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) \neq \phi$$

for every finite number of open sets $a_{\lambda_\nu} \in U_{\lambda_\nu} \in \mathfrak{N}_{\lambda_\nu}$, $\lambda_\nu \in \Lambda$ $(\nu=1,2,\cdots,n)$, we can find a point $x \in R$ for which $\mathfrak{a}_\lambda(x)=a_\lambda$ for every $\lambda \in \Lambda$.

In the sequel, we consider generalization of this theorem in the case where S_λ $(\lambda \in \Lambda)$ are merely compact.

Theorem 2. *If all S_λ $(\lambda \in \Lambda)$ are compact and for any system of points $a_\lambda \in S_\lambda$ $(\lambda \in \Lambda)$ subject to the condition* (F), *we can find a point $x \in R$ for which $\mathfrak{a}_\lambda(x) \in \{a_\lambda\}^-$ for every $\lambda \in \Lambda$, then the weak topology of R is compact.*

Proof. Let K be a maximal system of sets of R subject to the condition (I) $\bigcap_{\nu=1}^{n} K_\nu \neq \phi$ for every finite number of sets $K_\nu \in \mathfrak{K}$ $(\nu=1,2,\cdots,n)$. We see easily then that $A \frown K \neq \phi$ for all $K \in \mathfrak{K}$ implies $A \in \mathfrak{K}$, and L, $K \in \mathfrak{K}$ implies $L \frown K \in \mathfrak{K}$. For any $\lambda \in \Lambda$, we have obviously $\bigcap_{\nu=1}^{n} \mathfrak{a}_\lambda(K_\nu) \neq \phi$ for every finite number of sets $K_\nu \in \mathfrak{K}$ $(\nu=1,2,\cdots,n)$, and hence $\bigcap_{K \in \mathfrak{K}} \mathfrak{a}_\lambda(K)^- \neq \phi$, because S_λ is compact by assumption. For a point $a_\lambda \in \bigcap_{K \in \mathfrak{K}} a_\lambda(K)^-$, we have

$$\mathfrak{a}_\lambda^{-1}(U) \in \mathfrak{K} \quad \text{for } a_\lambda \in U \in \mathfrak{N}_\lambda,$$

because for $a_\lambda \in U \in \mathfrak{N}_\lambda$, $K \in \mathfrak{K}$ we have obviously

$$\mathfrak{a}_\lambda(K \frown \mathfrak{a}_\lambda^{-1}(U)) = \mathfrak{a}_\lambda(K) \frown U \neq \phi$$

which yields $K \frown \mathfrak{a}_\lambda^{-1}(U) \neq \phi$. Therefore the system of points a_λ $(\lambda \in \Lambda)$ satisfies the condition (F), and hence we can find a point $x \in R$ by assumption such that $\mathfrak{a}_\lambda(x) \in \{a_\lambda\}^-$ for every $\lambda \in \Lambda$. For such a point $x \in R$, we have obviously $\mathfrak{a}_\lambda(x) \in \bigcap_{K \in \mathfrak{K}} \mathfrak{a}_\lambda(K)^-$, and consequently $\mathfrak{a}_\lambda^{-1}(U) \in \mathfrak{K}$ for $\mathfrak{a}_\lambda(x) \in U \in \mathfrak{N}_\lambda$, as proved just above. Therefore we have