99. A New Characterization of Paracompactness

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1959)

The present note deals with another characterization of paracompactness of regular spaces and linearly ordered spaces. Our result concerning regular spaces is closely related to that of Kelley [2, p. 156], which asserts that if X is a regular space, then X is paracompact if and only if each open covering of X is even. Moreover our result concerning linearly ordered spaces is a generalization of that of Gillman and Henriksen [1] which asserts that a linearly ordered Q-space is paracompact.

1. For an open covering $\mathfrak{U}=\{G_\gamma \mid \gamma \ni \Gamma\}$ of a T-space X and a neighborhood U of the diagonal Δ of the product space $X\times X$, let A_U be the closure of the set of all points x such that $U(x)$ is not contained in every member G_γ of $\mathfrak{U}$, where $U(x)=\{y \mid (x, y)\in U\}$. It is clear that if $U\subset V$, then $A_U\subset A_V$. Set $H_U=A_U^c$. Then H_U is an open set of X, and if $U\subset V$, then $H_U\supset H_V$. Now let $\mathfrak{F}$ be the family of all the neighborhoods of the diagonal Δ of $X\times X$. Then we have the following

Lemma. *If X is a regular space, then $\{H_U \mid U\in\mathfrak{F}\}$ is an open covering of X.*

Proof. If there is an A_U such that $A_U=\phi$, then for such A_U we have $H_U=A_U^c=X$. Now suppose that $A_U\neq\phi$ for every $U\in\mathfrak{F}$. For any point x of X there is a G_γ such that $x\in G_\gamma$, and, since X is regular, there are open sets H and K such that

$$G_\gamma\supset\overline{H}\supset H\supset\overline{K}\supset K\ni x.$$

Let us put $U=(H\times H)\cup(K^c\times K^c)$. Then U is a neighborhood of the diagonal Δ, and $U(x)=H\subset G_\gamma$. Moreover for any point y of K, $U(y)=H\subset G_\gamma$. Therefore x is contained in $H_U=A_U^c$. Thus $\{H_U \mid U\in\mathfrak{F}\}$ is an open covering of X. This completes the proof of the lemma.

In case X is regular, we call $\tilde{\mathfrak{U}}=\{H_U \mid U\in\mathfrak{F}\}$ an open covering of X derived from an open covering $\mathfrak{U}$ of X.

Theorem 1. *If X is a regular space, then the following statements are equivalent:*

(1) *X is paracompact.*

(2) *Every open covering $\tilde{\mathfrak{U}}$ of X derived from any open covering $\mathfrak{U}$ of X has a finite subcovering.*

Proof. (1)$\rightarrow$(2). Since X is paracompact, any open covering $\mathfrak{U}$ is even, that is, there is a $U_0\in\mathfrak{F}$ such that for each x $U_0(x)$ is contained in some member of $\mathfrak{U}$. This shows that $A_{U_0}=\phi$, i.e. $H_{U_0}=X$. Hence