96. Some Characterizations of Fourier Transforms

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In the following we shall show that the Fourier cosine transform and the Fourier exponential transform are characterized by some of their properties.

At first we shall prove a number-theoretical lemma. Let

$$p_1 < p_2 < p_3 < \cdots$$

be the all prime numbers and $\mu_\nu(n)$ a function defined at every natural number such that $\mu_\nu(n)=\mu(n)$, if every prime divisor of n is one of $p_1, p_2, \cdots, p_\nu$, and $\mu_\nu(n)=0$ otherwise.

Lemma. *Let $f(n)$ be a function defined at every non-negative integer and $\sum_{n=0}^{\infty} f(n)$ absolutely convergent. Let us denote*

$$F(m)=\sum_{n=0}^{\infty} f(mn)$$

for every natural number m. Then

$$f_\nu(m)=\sum_{n=1}^{\infty} \mu_\nu(n)\ F(mn)$$

converges to $f(m)$ as $\nu\to\infty$.

Proof. We have

$$f_\nu(m)=\sum_{n=0}^{\infty} f(mn)\sum_{d\mid n} \mu_\nu(d)$$

and

$$\sum_{d\mid n} \mu_\nu(d)=\begin{cases}1, & (n, p_1 p_2\cdots p_\nu)=1,\\ 0 & \text{otherwise,}\end{cases}$$

therefore

$$f_\nu(m)=\sum f(mn),$$

where n ranges over all positive integers prime to $p_1 p_2\cdots p_\nu$. Then

$$|f(m)-f_\nu(m)|\leqq\sum_{n>p_\nu} |f(mn)|$$

and the right hand side of this inequality tends to 0 as $\nu\to\infty$. Q. E. D.

By $\mathfrak{D}$ we denote the family of all $C^\infty(R)$-functions with compact carrier. For a given continuous function $F(x)$ we denote

$$F\varphi(x)=\int_{-\infty}^{\infty} F(xt)\varphi(t)dt, \qquad \varphi\in\mathfrak{D}.$$

Theorem 1. *Let an even function $C(x)$ be the second derivative of a bounded function, and*

$$\sum_{n=-\infty}^{\infty} C\varphi(n)=\sum_{n=-\infty}^{\infty} \varphi(n) \tag{1}$$

for all $\varphi\in\mathfrak{D}$. Then