94. *A Tauberian Theorem for Fourier Series*

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1. Let $\varphi(t)$ be an even function, integrable in Lebesgue sense, periodic of period 2π, and let

$$\varphi(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt$$

and

$$s_n=\frac{1}{2}a_0+\sum_{\nu=1}^{n}a_\nu.$$

Hardy and Littlewood [1] have proved that if

$$\int_0^t|\varphi(u)|\,du=o\left(t\Big/\log\frac{1}{t}\right)\qquad(t\to 0),$$

and if for some positive δ

$$a_n>-An^{-\delta},\quad A>0,$$

then

$$s_n\to 0\quad\text{as}\quad n\to\infty.$$

In this paper we shall prove a converse

THEOREM. If $\sum a_n$ is summable to zero in Abel sense, and

$$(1)\qquad \sum_{\nu=n}^{2n}|a_\nu|=o(1/\log n)\qquad(n\to\infty),$$

and if for some positive ρ,

$$(2)\qquad \varphi'(t)>-At^{-\rho}\qquad(0<t<t_0),$$

where A is a positive constant independent of t, then

$$\varphi(t)\to 0\qquad(t\to 0).$$

2. Proof of the theorem. We require a

LEMMA. If $\sum u_n$ is summable in Abel sense, and if

$$u_{n+1}+u_{n+2}+\cdots+u_{n+\nu}>-K\quad(\nu=1,2,\cdots,n),$$

where K is a positive constant, then the series $\sum u_n$ converges to the same sum.

This is Lemma 2, slightly modified, of Szász [2].

For the proof of our Theorem, using the argument in Yano [3], we begin with the identities

$$(3)\qquad \varphi(t)=\frac{1}{h}\int_0^h\varphi(t+u)du-\frac{1}{h}\int_0^h[\varphi(t+u)-\varphi(t)]du$$

and

$$(4)\qquad \varphi(t)=\frac{1}{h}\int_0^h\varphi(t-u)du+\frac{1}{h}\int_0^h[\varphi(t)-\varphi(t-u)]du,$$