93. On the Thue-Siegel-Roth Theorem. I

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1. The main object of this note is to show that the Thue-Siegel-Roth theorem can somewhat be refined when the field of reference is an imaginary quadratic number field. The Thue-Siegel-Roth theorem [1] is

Theorem 1. *Let K be an algebraic number field of finite degree and let α be algebraic of degree at least 2 over K. Then for each $\kappa>2$, the inequality*

$$|\alpha-\xi|<(H(\xi))^{-\kappa} \tag{1}$$

has only a finite number of solutions ξ in K.

Here $H(\xi)$ denotes the height of ξ, the maximum of the absolute values of the coefficients in the primitive irreducible equation with rational integral coefficients of which ξ is a zero, while we designate by $M(\xi)$ the absolute value of the highest coefficient in that equation for ξ.

Since an algebraic number field K of finite degree has only finitely many subfields and every element of K is a primitive number of some one of its subfields, in order to establish Theorem 1 it is enough to prove that for each $\kappa>2$, the inequality (1) is satisfied by only finitely many primitive numbers ξ in K. In this respect the following theorem will be of some interest:

Theorem 2. *Let α be any non-zero algebraic number and let K be an imaginary quadratic number field. If the inequality*

$$|\alpha-\xi|<(M(\xi))^{-\kappa} \tag{2}$$

is satisfied by infinitely many primitive numbers ξ in K, then $\kappa\leqq 1$.

It is clear that $M(\xi)\leqq H(\xi)$ for any fixed ξ and $M(\xi)=1$ for any integral ξ. From this result one can deduce at once the following

Theorem 3. *Let α and K be as in Theorem 2. Then for each $\nu>2$, the inequality*

$$0<\left|\alpha-\frac{p}{q}\right|<\frac{1}{|q|^{\nu}} \tag{3}$$

has only a finite number of integer solutions p, q $(q\neq 0)$ in K.

If, in (3), p and q $(q\neq 0)$ are restricted to be rational integers, Theorem 3 reduces to a recent result of K. F. Roth [3], and we may exclude this rational case. Then the fraction p/q with integers p, q $(q\neq 0)$ in K is a primitive number ξ in K, and, for any representation $\xi=p'/q'$ of the number ξ with integers p', q' $(q'\neq 0)$ in K, it satisfies