128. A Remark on a Theorem of J. P. Serre

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1959)

1. The purpose of this note is to prove the following

Theorem. *Let p be an odd prime, and let X be an arcwise- and simply-connected topological space satisfying*

i) $H_i(X, Z)$ *is finitely generated for all* $i>0$,

ii) $H_i(X, Z_p)=0$ *for all sufficiently large* i,

iii) $H_i(X, Z_p)\neq 0$ *for some* $i>0$.

Then there exist infinitely many values of i such that $\pi_i(X)$ has a subgroup isomorphic to Z or Z_p.

If we apply this theorem to $X=S^n$, a sphere of dimension $n\geq 2$, we obtain the result that for each S^n there exist infinitely many values of i such that the p-component of $\pi_i(S^n)$ is not zero and thus solve affirmatively Problem 12 of W. S. Massey.[1]

The above theorem was proved by J. P. Serre in the case $p=2$.[2] Our method of proof is a modification of that of Serre by using the results on $H_*(\pi, n; Z_p)$ due to H. Cartan.[3]

Throughout this note p is assumed to denote an odd prime.

2. **Lemma.** Let $n\geq 1$, and let π be a finitely generated abelian group. Then

i) $\vartheta(\pi, n; t)=\sum_{i=0}^{\infty}(\dim H_i(\pi, n; Z_p))t^i$ converges in the disk $|t|<1$.

ii) Setting

$$\varphi(\pi, n; x)=\log_p(\vartheta(\pi, n; 1-p^{-x})) \quad \text{for } 0\leq x<+\infty,$$

we have the following valuations. ($f(x)\sim g(x)$ means $\lim_{x\to+\infty} f(x)/g(x)=1$.)

$$\varphi(Z_{p^f}, n; x)\sim x^n/n!, \quad \varphi(Z, n; x)\sim\begin{cases} x^{n-1}/(n-1)! & \text{for } n\geq 2,\\ \log_p 2 & \text{for } n=1,\end{cases}$$

$$\varphi(Z_{q^f}, n; x)=0, \quad \text{where } q^f \text{ is a power of a prime } q(\neq p).$$

Proof of Lemma. We prove i) first. By the Künneth's relation $\vartheta(\pi+\pi', n; t)=\vartheta(\pi, n; t)\vartheta(\pi', n; t)$ for any finitely generated abelian groups π and π', it suffices to prove i) when $\pi=Z_{p^f}$ or Z or Z_{q^f}, where p^f and q^f mean the same as in ii). The case $\pi=Z_{q^f}$ is trivial, since $\vartheta(Z_{q^f}, n; t)=1$. The following expression (1) of $\vartheta(Z_{p^f}, n; t)$ is

1) W. S. Massey: Some problems in algebraic topology and the theory of fibre bundles, Ann. Math., **62**, 327-359 (1955).

According to this article, Problem 12 was also solved affirmatively by I. M. James.

2) J. P. Serre: Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, Comment. Math. Helv., **27**, 198-232, Theorem 10 (1953).

3) H. Cartan: Séminaire H. Cartan, E. N. S., 1954-1955.