123. On Monotone Solutions of Differential Equations

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1959)

In a recent note [1], Professor Iséki proved the following theorem, which we will formulate for one single differential equation: *If the functions* $P(t)$, $Q(t)$ *are defined and absolutely integrable on an interval* $[a, +\infty)$, *then any monotone increasing solution* $x(t)$ *of*

$$\frac{dx}{dt}=P(t)x+Q(t)$$

is bounded on that interval.

It is natural to ask whether a similar theorem may hold for an equation

$$\frac{dx}{dt}=\sum_{m=0}^{n} p_m(t)x^m \tag{1}$$

with suitable conditions on the coefficients $p_m(t)$. It is immediately seen that if the leading coefficient $p_n(t)$ is integrable, no similar result can hold, since

$$\frac{dx}{dt}=\frac{1}{t^2}x^2$$

has the unbounded solution $x=t$. So one may try to get the desired result from the opposite condition: $P_n(t)$ not integrable on $[a, +\infty)$, since the example

$$\frac{dx}{dt}=\frac{1}{t}x^2$$

has the monotone solution $x=-(\log t)^{-1}$, bounded by 0.

It turns out that this special situation is the general one, since we have

Theorem 1. *If the functions* $p_m(t)$, $m=0,\cdots,n$ *are defined on an interval* $[a, +\infty)$, *and if*

i) $\displaystyle\int_a^\infty p_n(u)du=+\infty,\quad p_n(t)\geq 0,\qquad (t\geq T_0)$

ii) $p_m(t)\geq 0,\quad 0\leq m\leq n-1,\qquad (t\geq T_0)$

then any monotone increasing solution $x(t)$ *of* (1)

a) *either is bounded by zero*

b) *or* $\displaystyle\lim_{t\to\infty}\frac{x(t)}{\int_T^t p_n(u)du}=\infty.$

Proof. By hypothesis i), $p_n(t)$ is positive for $t>T_0$. Now suppose that there is a T (it may be taken $>T_0$) with $x(T)>0$, $x(t)$ being a