# 122. On Finite Dimensional Quasi-norm Spaces

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In this Note, we shall consider a *finite dimensional quasi-norm space* $E$*) *of order* $r$. Suppose that the dimension of $E$ is $n$ and let $e_1, e_2, \cdots, e_n$ be the bases of $E$. Then any element $x$ of $E$ may be written in the form

$$x=\lambda_1 e_1+\lambda_2 e_2+\cdots+\lambda_n e_n.$$

Let $\{x_m\}$ be a sequence of $E$, and let

$$x_m=\sum_{i=1}^{n}\lambda_i^m e_i.$$

If $\lambda_i^m \to \lambda_i$ $(m\to\infty)$ for every $i$,

$$\|x_m-x\|=\left\|\sum_{i=1}^{n}(\lambda_i^m-\lambda_i)e_i\right\|\leq\sum_{i=1}^{n}\|(\lambda_i^m-\lambda_i)e_i\|$$
$$\leq|\lambda_i^m-\lambda_i|^r\sum_{i=1}^{n}\|e_i\|\to 0 \quad (m\to\infty).$$

Hence we have $x_m\to x$ $(m\to\infty)$.

Now we shall prove the following

*Lemma. For any element* $x=\sum_{i=1}^{n}\lambda_i e_i$ *of* $E$, *there is a positive number* $H$ *such that*

$$|\lambda_i|^r\leq H\|x\|,$$

*where* $H$ *depends on the base* $e_i$ *of* $E$.

Proof. Let $S$ be the unit sphere of $n$-dimensional space $R^n$. For $\Xi=(\lambda_1,\cdots,\lambda_n)$ we put $x(\Xi)=\sum_{i=1}^{n}\lambda_i e_i$, the linear independence of $e_i$ and $\sum_{i=1}^{n}\lambda_i^2=1$ imply $x(\Xi)\neq 0$. As mentioned above, $\Xi^m\to\Xi$ $(m\to\infty)$ in $R^n$ implies $x(\Xi^m)\to x(\Xi)$. Hence $x(\Xi)$ is continuous on the compact set $S$. Therefore we have $m=\underset{\Xi\in S}{\mathrm{Min}}\|x(\Xi)\|>0$.

Let $H=\dfrac{1}{m}$, and take a non-zero element $x=\sum_{i=1}^{n}x_i e_i$ of $E$

$$x'=\frac{1}{\sqrt{\sum_{i=1}^{n}\lambda_i^2}}x=\sum_{i=1}^{n}\mu_i x_i,$$

where

$$\mu_K=\frac{\lambda_K}{\sqrt{\sum_{i=1}^{n}\lambda_i^2}}.$$

From $\sum_{i=1}^{n}\mu_i^2=1$, we have $\|x'\|\geq m$. Hence

*) For details, see T. Konda [1], M. Pavel [2], and S. Rolewicz [3].