120. On the Thue-Siegel-Roth Theorem. II

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1. This is a continuation of a previous note under the same title [6]. In the following we shall be concerned with some further results closely related to the Thue-Siegel-Roth theorem on the approximability of an algebraic number by other algebraic numbers.

2. The Thue-Siegel-Roth theorem [2] is an immediate consequence of the following

Theorem 1. *Let α be any algebraic number other than zero and let K be an algebraic number field of finite degree over the rationals. If the inequality*

$$|\alpha-\xi|<(H(\xi))^{-\kappa} \tag{1}$$

is satisfied by infinitely many primitive numbers ξ in K, then

$$\kappa\leqq\begin{cases} 2 & \text{when } K \text{ is real,} \\ 1 & \text{when } K \text{ is complex.}^{1)} \end{cases} \tag{2}$$

Moreover, when K is the rational number field or an imaginary quadratic number field, $H(\xi)$ in (1) *can be replaced by $M(\xi)$ and the bound* (2) *for κ is best possible.*

For the definition of $H(\xi)$ and $M(\xi)$ we refer to [6, §1]. The first part of Theorem 1 is easily seen from W. J. LeVeque's proof [2] of the Thue-Siegel-Roth theorem, and the second part is a well-known theorem due to K. F. Roth [5] when K is the rational number field, and Theorem 2 in [6] when K is an imaginary quadratic field. We note that it is impossible, in general, to replace $H(\xi)$ in (1) by $M(\xi)$.

3. Let K be an algebraic number field. A non-zero integer of K is said to be *prime in K* if the principal ideal generated by the integer is a prime ideal in K. The associates of a number in K will be identified with the number itself.

Theorem 2. *Let α be any non-zero algebraic number and let K be an imaginary quadratic number field. Let $u_1,\cdots,u_s$, $v_1,\cdots,v_t$ be a finite set of distinct integers of K, each being supposed to be prime in K. Let μ, ν, c be real numbers satisfying*

$$0\leqq\mu\leqq1,\quad 0\leqq\nu\leqq1,\quad c>0.$$

Let p, q be integers in K of the form

$$p=p^*u_1^{a_1}\cdots u_s^{a_s},\quad q=q^*v_1^{b_1}\cdots v_t^{b_t},$$

where $a_1,\cdots,a_s,b_1,\cdots,b_t$ are non-negative rational integers and p^*, q^* are integers of K such that

1) A field is complex if it is not a real field.