133. On Quasi-normed Spaces. II

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In this paper, we shall consider some theorems in (QN)-spaces. For definitions and notations, see my paper [2], M. Pavel [3] and S. Rolewicz [4].

First of all, we shall prove the following

Lemma. *If L is a proper subspace of the (QN)-space E with the power r, then for any $\varepsilon>0$ and the element y of E such that $\|y\|=1$, every element x of L satisfies the inequality $\|x-y\|>1-\varepsilon$.*

Proof. We take an element $y_0\in E$ such that $y_0\notin L$ and put $d=\inf_{x\in L}\|y_0-x\|$. Then we have $d>0$. For any $\eta>0$, we select also an element $x_0\in L$ such that $d\leq\|y_0-x_0\|<d+\eta$. The element $y=\dfrac{y_0-x_0}{\|y_0-x_0\|^{1/r}}$ is not contained in L, for if y is in L then y_0 must be in L. Moreover $\|y\|=1$ and for any $x\in L$, $x'=x_0+\|y_0-x_0\|^{1/r}x$ and $x'\in L$, we have

$$\|y-x\|=\left\|\frac{y_0-x_0}{\|y_0-x_0\|^{1/r}}-x\right\|=\frac{1}{\|y_0-x_0\|}\|y_0-x'\|$$

$$>\frac{1}{d+\eta}\|y_0-x'\|\geq\frac{d}{d+\eta}=1-\frac{\eta}{d+\eta}.$$

Since η is arbitrary, we can take η such that $\dfrac{\eta}{d+\eta}<\varepsilon$ and $\eta>0$. Thus we have the desired result.

Theorem I. *A subspace L of a (QN)-space E with the power r is a finite dimensional space if and only if any bounded subset of L is compact.* (For Banach space, see [1, pp. 76-78].)

Proof. Necessary. Let L be n-dimensional. Any element $x\in L$ is of form $x=\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n$ with a base $\{x_i\}$ of L for $i=1,2,\cdots,n$.

Let $\{y_k\}$ be a bounded sequence in L, then we can write $y_k=\lambda_1^{(k)}x_1+\cdots+\lambda_n^{(k)}x_n$ for $k=1,2,\cdots$. By the boundness of $\{y_k\}$ there exists M such that $\|y_k\|\leq M$ for $k=1,2,\cdots$ and it may be proved that the sum $|\lambda_1^{(k)}|^r+\cdots+|\lambda_n^{(k)}|^r$ is bounded. For if the sum is not bounded, then there exists a sequence of indexes $K_1,K_2,\cdots$ such that

$$|\lambda_1^{(k_m)}|^r+|\lambda_2^{(k_m)}|^r+\cdots+|\lambda_n^{(k_m)}|^r=c_m\geq m.$$

Let $y^*_{k_m}=\dfrac{1}{c_m^{1/r}}y_{k_m}$, then we have

$$\|y^*_{k_m}\|=\frac{1}{c_m}\|y_{k_m}\|\leq\frac{1}{c_m}M\leq\frac{M}{m}$$