132. Some Notes on Cesàro Summation

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In this paper we shall establish two lemmas concerning the Cesàro summability of Fourier series. Of these, Theorem 1 is closely related to the result of Chandrasekharan and Szász [2, Theorem 5]. And Theorem 2 is concerned with the estimation of the principal part of Fejér kernels.

1. THEOREM 1. If $\varphi(t)\in L$ in $0\leqq t\leqq t_0$, and $r>0$, $\delta>0$, and q be arbitrary, then

$$\Phi_r(t)\equiv\frac{1}{\Gamma(r)}\int_0^t(t-u)^{r-1}\varphi(u)du=o(t^q)\qquad(t\to 0)\tag{1.1}$$

is equivalent to

$$\Phi_r^\delta(t)\equiv\frac{1}{\Gamma(r)}\int_0^t(t-u)^{r-1}u^\delta\varphi(u)du=o(t^{q+\delta})\qquad(t\to 0).\tag{1.2}$$

Letting

$$\varphi_r^\delta(t)=\frac{\Gamma(r+\delta+1)}{\Gamma(\delta+1)}t^{-(r+\delta)}\Phi_r^\delta(t)\qquad(\delta\geqq 0),$$

and $\varphi_r(t)=\varphi_r^0(t)$, we have the following

COROLLARY 1. Let $\varphi(t)\in L$ in $(0,t_0)$, and $r>0$, $\delta>0$, and q be arbitrary. Then

$$\varphi_r(t)=s+o(t^{q-r})\qquad(t\to 0)$$

is equivalent to

$$\varphi_r^\delta(t)=s+o(t^{q-r})\qquad(t\to 0),$$

where s is a constant independent of t.

Concerning this corollary, cf. loc. cit. [2].

We need two lemmas:

LEMMA 1. Theorem 1 holds when $\delta=k$, where k is a positive integer.

This is Lemma 3 in the paper [3], but for the sake of completeness we prove it. We first consider the case $k=1$. Observe now that

$$\Phi_r^1(t)=t\Phi_r(t)-r\Phi_{r+1}(t),\tag{1.3}$$

and that necessarily, since $r>0$,

$$\Phi_{r+1}(t)=o(t^r).\tag{1.4}$$

If $q>-1$, then (1.1) implies

$$\Phi_{r+1}(t)=o(t^{q+1}),\tag{1.5}$$

and then by (1.3),

$$\Phi_r^1(t)=o(t^{q+1}),\tag{1.6}$$

which follows from (1.1) still when $q\leqq -1$, by (1.3) and (1.4).