7. A Class of Quasi-normed Spaces

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A non-Archimedean normed space was considered by A. F. Monna [2] and I. S. Cohen [1]. We shall consider a non-Archimedean quasi-normed space. By a non-Archimedean quasi-normed space with power r, we shall mean a linear space E over a commutative field K such that to every x of E there corresponds a real number $||x||$ such that

1) $||x||>0$ for $x\neq 0$.

2) $||x+y||\leq \mathrm{Max}(||x||, ||y||)$ for x, y of E.

3) $||\lambda x||=|\lambda|^r ||x||$ for $\lambda\in K$ and $x\in E$,

where $|\lambda|$ is a non-Archimedean valuation of K.

It is clear that the function $d(x,y)=||x-y||$ defines a metric on the space E.

Now we shall show the following

Proposition 1. *Let F be a closed linear subspace of E. If the sequence $\{x_n+a_n y\}$ converges in E for a fixed element $y\bar{\in} F$ and $x_n\in F$, $a_n\in K$, then $\{x_n\}$ and $\{a_n\}$ are convergent.*

Proof. We shall prove that $x_n+a_n y\to 0$ $(n\to\infty)$ implies $a_n\to 0$. Suppose that a_n does not converge, then there is a subsequence $\{a_{k_n}\}$ such that $|a_{k_n}|\geq\varepsilon$ for some positive number ε. Hence

$$\begin{aligned}||a_{k_n}^{-1}(x_{k_n}+a_{k_n}y)&=|a_{k_n}^{-1}|^r ||x_{k_n}+a_{k_n}y||\\ &\leq\varepsilon^{-r}||x_{k_n}+a_{k_n}y||\end{aligned}$$

implies $a_{k_n}^{-1}(x_{k_n}+a_{k_n}y)=a_{k_n}^{-1}x_{k_n}+y\to 0$. From $a_{k_n}^{-}x_{k_n}\in F$ and closedness of F, we have $y\in F$.

For the general case, from the existence of $\lim_{n\to\infty}(x_n+a_n y)$, we have $(x_{n+1}-x_n)+(a_{n+1}-a_n)y\to 0$, and we can conclude $a_{n+1}-a_n\to 0$. Hence, by a property of non-Archimedean valuation $\{a_n\}$ is a fundamental sequence and we can find the limit of $\{a_n\}$.

Proposition 2. *Any finite dimensional subspace of E is closed.*

Proof. If F is a closed linear subspace of E, then $F+[y]$ is closed, where $F+[y]$ denotes the minimal linear subspace generated by F and y, i.e. $F+[y]=\{x+\alpha y, x\in F, \alpha\in K\}$. Suppose that $y\bar{\in} F$, and $\{x_n+\alpha_n y\}(x_n\in F, \alpha_n\in K)$ converges to an element of E, then, by Proposition 1, x_n converges to an element x of E, and α_n converges to α of E. Since F is closed, the element x is contained in F. Therefore $x+\alpha y\in F+[y]$, and $F+[y]$ is closed.

Theorem 1. *Any finite dimensional non-Archimedean normed space E is topologically Euclidean and is complete.*