6. On Some Properties of Group Characters

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Let $\mathfrak{G}$ be a group of finite order and let p be a fixed prime number. An element is called a p-element of $\mathfrak{G}$ if its order is a power of p. An arbitrary element G of $\mathfrak{G}$ can be written uniquely as a product PR of two commutative elements where P is a p-element, while R is a p-regular element, i.e. an element whose order is prime to p. We shall call P the p-factor of G and R the p-regular factor of G. We define the section $\mathfrak{S}(P)$ of a p-element P as the set of all elements of $\mathfrak{G}$ whose p-factor is conjugate to P in $\mathfrak{G}$. Let $\mathfrak{K}_\nu$ be a class of conjugate elements which contains an element whose p-factor is P. Then $\mathfrak{S}(P)$ is the union of these classes $\mathfrak{K}_\nu$. Let $P_1=1, P_2,\cdots,P_h$ be a system of p-elements such that they all lie in different classes of conjugate elements, but that every p-element is conjugate to one of them. Then all elements of $\mathfrak{G}$ are distributed into h sections $\mathfrak{S}(P_i)$.

We consider the representations of $\mathfrak{G}$ in the field of all complex numbers. Let $\chi_1, \chi_2,\cdots,\chi_n$ be the distinct irreducible characters of $\mathfrak{G}$. Then the χ_i are distributed into a certain number of blocks $B_1, B_2,\cdots,B_t$. We denote by $\bar{a}$ the conjugate of a complex number a. Then $\bar{\chi}_i(G) =\chi_i(G^{-1})$. In [1] the following theorem has been stated without proof:

Let B be a block of $\mathfrak{G}$. If the elements G and H of $\mathfrak{G}$ belong to different sections of $\mathfrak{G}$, then

$$\sum \chi_i(G)\bar{\chi}_i(H)=0 \tag{1}$$

where the sum extends over all $\chi_i \in B$.

Recently the proof of this theorem was given in [2]. In this note, corresponding to the above theorem, we shall prove the following

Theorem 1. *Let $\mathfrak{S}(P)$ be a section of $\mathfrak{G}$. If the characters χ_i and χ_j belong to different blocks, then*

$$\sum{}' \chi_i(G)\bar{\chi}_j(G)=0$$

where the sum extends over all $G\in\mathfrak{S}(P)$.

As a consequence of Theorem 1, some new results are also obtained.

1. Let $\mathfrak{K}_\nu$ $(\nu=1,2,\cdots,n)$ be the classes of conjugate elements in $\mathfrak{G}$ and let G_ν be a representative of $\mathfrak{K}_\nu$. We shall first prove the following

Lemma. *If $\sum_{\nu=1}^{n} a_\nu\chi_i(G_\nu)=0$ for all $\chi_i\in B$, then $\sum_\alpha{}' a_\alpha\chi_i(G_\alpha)=0$ where the sum extends over all $\mathfrak{K}_\alpha\in\mathfrak{S}(P)$.*

Proof. Let $\mathfrak{K}_\beta$ be a class belonging to $\mathfrak{S}(P)$. We multiply by