5. A Remark on Quasi-Frobenius Rings

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1. Throughout this note A will denote a ring with unit satisfying the minimum conditions for left and right ideals.

We shall consider the following conditions:

Condition (P). Let M be a module. If two submodules N_1 and N_2 are isomorphic, then the two residue modules M/N_1 and M/N_2 are isomorphic, too.

Condition (L_n). Let A be a ring and $A^{(n)}$ the direct sum of n isomorphic copies of the left A-module A. Then the module $A^{(n)}$ satisfies (P).

Condition (R_n)=the right-left symmetry of (L_n).

The following facts are known:

(1) If A is quasi-Frobenius (in short, QF), then A satisfies (L_n) and (R_n) for $n=1, 2, \cdots$ [3, Corollary 4.3]. (See also [4, Theorem 2.3].)

(2) Conversely, if A satisfies (L_n) and (R_n) for every natural number n, then A is QF [3, Theorem 4.4].

(3) If A is an algebra over an algebraically closed field, and if A satisfies (L_1) (or (R_1)), then A is QF [2, Theorem 3].

(4) There exists an algebra A satisfying (L_1) and (R_1) which is not QF [2, Remark].

Lemma 1. Let M be a module of finite length, and D a direct summand of M. If M satisfies (P), D also satisfies (P).

Proof. Let N_1 and N_2 be mutually isomorphic submodules of D, and let $M=D\oplus D'$. Then, $(D/N_1)\oplus D'\simeq M/N_1\simeq M/N_2\simeq(D/N_2)\oplus D'$ by assumption. Therefore $D/N_1\simeq D/N_2$ by the Krull-Remak-Schmidt theorem, as desired.

From this lemma and the proof of [3, Theorem 4.4] it follows that every ring A satisfying (L_2) *and* (R_2) is QF. The purpose of the present note is to show the following

Theorem 2. *Let A be a ring, and B a left A-module. Suppose that* (1) *A is a direct summand of B, and* (2) *for every indecomposable summand Ae_k of A, B contains a direct summand which is the direct sum of two isomorphic copies of Ae_k. If B satisfies* (P), *then A is* QF.

As immediate consequences we obtain

Corollary 3. If A satisfies (L_2) (*or* (R_2)) then A is QF.

Corollary 4. Let A_2 be the total matrix ring of degree 2 over