21. A Note on (E.R.)integral and Fourier Series

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1. *Introduction and notations.* In this note we shall consider (E.R.)integral of rather special type.[1] Let $\mathcal{E}$ be the set of all step functions defined on the finite closed interval $[a, b]$. In $\mathcal{E}$ we shall introduce a topology and a rank so that $\mathcal{E}$ becomes a ranked space. Let F be the closed set in $[a, b]$, ν be an integer and f be a function in $\mathcal{E}$, then we shall define a neighbourhood of f, $V(F, \nu; f)$ as the set of all functions $g \in \mathcal{E}$ such that there exist in $\mathcal{E}$ functions $p(x)$ and $r(x): g(x)-f(x)=p(x)+r(x)$ which satisfy the following conditions:

[1] $r(x)$ is zero for all x in F,

[2] we have $\int_a^b |p(x)|\,dx<2^{-\nu}$,

[3_1] for every $c, d: a\leqq c<d\leqq b$ we have $\left|\int_c^d r(x)dx\right|<2^{-\nu}$.[1]

A neighbourhood $V(F, \nu; f)$ is called *of rank* ν, if we have $\text{mes}\{[a, b]-F\}<2^{-\nu}$. A sequence of neighbourhoods $\{u_n\}=\{V(F_n, \nu_n; f_n)\}$ is called fundamental sequence, if we have

(1) $u_0 \supseteq u_1 \supseteq \cdots \supseteq u_n \supseteq \cdots$, the rank of u_n is ν_n,

(2) $\nu_0 \leqq \nu_1 \leqq \cdots \leqq \nu_n \leqq \cdots$,

(3) $f_{2n}=f_{2n+1}$, $\nu_{2n}<\nu_{2n+1}$, $n=0, 1, 2, \cdots$

Further we shall add the conditions:

(1*) the sequence $\{u_n\}$ has the property P, that is, there exists a function $\phi(n)$ $(n=0, 1, 2, \cdots)$ such that $\phi(n)>0$ for $n=0, 1, 2, \cdots$, $\lim_{n\to\infty}\phi(n)=0$, and for every measurable set E contained in $[a, b]$ whose measure is less than $\text{mes}\{[a, b]-F_n\}$, we have

$$\int_E |f_n(x)|\,dx \leqq \phi(n). \tag{1.1}$$

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1) The investigation of (E.R.)integral originates from the note of Prof. K. Kunugi: Application de la méthode des espaces rangés à la théorie de l'intégration. I, Proc. Japan Acad., **32**, 215–220 (1956). In original note the condition [3_1] is weaker than the present one, that is,

[3] we have $\left|\int_a^b r(x)dx\right|<2^{-\nu}$.

While the condition [3_1] was first considered by Dr. S. Nakanishi: Sur la dérivation de l'intégrale (E.R.)indéfinie. I, Proc. Japan Acad., **34**, 199–204 (1958), which makes the indefinite integral continuous.

In this present note we owe also the note of Prof. K. Kunugi: Sur une généralisation de l'intégrale, Fundamental and Applied Aspects of Mathematics, 1–30, Research Institute of Applied Electricity, Hokkaido University.