# 35. An Application of a Compact Normal Operator in Hilbert Spaces to the Theory of Functions

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In this paper we shall discuss the integration of a given function of a complex variable along a closed Jordan curve which encloses its denumerably infinite set of poles and its essential singularities, by making use of the properties of a compact normal operator in an abstract Hilbert space $\mathfrak{H}$ and of linear functionals with domain $\mathfrak{H}$.

Theorem 1. Let $f(\lambda)$ be holomorphic at all points of the closure $\overline{D}$ of a simply connected domain $D$ in the complex $\lambda$-plane, except at its poles $\{\lambda_n\}\in D$ tending to the point $\lambda=0$ interior to $D$ and at its non-isolated essential singularity $\lambda=0$.

If the principal part of the expansion of $f(\lambda)$ at any pole $\lambda_n$ is given by $\frac{\alpha_n}{\lambda-\lambda_n}$ and if $\sum_{n=1}^{\infty}|\alpha_n|<\infty$, then

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda=\sum_{n=1}^{\infty}\alpha_n,$$

where the complex curvilinear integration along the boundary $\partial D$ of $D$ is taken in the positive (anti-clockwise) direction.

Proof. Let $\{\varphi_n\}$ be an arbitrary complete orthonormal system in the abstract complex Hilbert space $\mathfrak{H}$ which is complete, separable and infinite dimensional, and let $E_n$ be the orthogonal projection of $\mathfrak{H}$ onto the subspace determined by $\varphi_n$.

If we now define $N$ by $N=\sum_{n=1}^{\infty}\lambda_n E_n$, it is easily verified that $N$ has the following properties:

1° the convergence of $\sum_{n=1}^{\infty}\lambda_n E_n$ is uniform, that is, $\left\|N-\sum_{n=1}^{p}\lambda_n E_n\right\|\to 0$, $(p\to\infty)$;

2° $\{\lambda_n\}$ is the point spectrum of $N$, and $E_n$ is the characteristic projection of $N$ corresponding to $\lambda_n$, $n=1,2,3,\cdots$;

3° $N$ is a compact normal operator in $\mathfrak{H}$ [1].

Since every linear continuous functional $L(y)$ on $\mathfrak{H}$ can be put in the form $L(y)=(y,x)$ where the generating element $x\in\mathfrak{H}$ is uniquely determined by the functional $L$ [4], from now on we shall denote by $L_x$ the functional $L$ associated with $x$.

Next we put

$$x=\sum_{n=1}^{\infty}\sqrt{\alpha_n}\,\varphi_n,\qquad \tilde{x}=\sum_{n=1}^{\infty}\sqrt{\overline{\alpha}_n}\,\varphi_n\qquad ((\sqrt{\alpha_n}\,\varphi_n,\ \sqrt{\overline{\alpha}_n}\,\varphi_n)=\alpha_n)$$