26. Note on Fractional Powers of Linear Operators

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In the preceding paper by K. Yosida,[1] it is shown that the fractional power A^{α}, $0<\alpha<1$, of a linear operator A in a Banach space X can be constructed whenever $-A$ is the infinitesimal generator of a strongly continuous, bounded semi-group $\{\exp(-tA)\}$, and that $-A^{\alpha}$ also generates a semi-group $\{\exp(-tA^{\alpha})\}$ which has an *analytic* extension in a sector containing the positive t-axis. In the present paper we shall give another proof of these results, together with some generalizations.

We consider linear operators in X which are not necessarily infinitesimal generators of semi-groups. For brevity we shall say that A is of type (ω, M)[2] if

i) A is densely defined[3] and closed, and

ii) the resolvent set of $-A$ contains the open sector $|\arg\lambda|<\pi-\omega$, $0<\omega<\pi$, and $\lambda(\lambda+A)^{-1}$ is uniformly bounded in each smaller sector $|\arg\lambda|<\pi-\omega-\varepsilon$, $\varepsilon>0$; in particular

$$\lambda\,\|(\lambda+A)^{-1}\|\leq M, \qquad \lambda>0. \tag{1}$$

As is well known, $-A$ is the infinitesimal generator of a strongly continuous contraction semi-group if and only if A is of type $(\pi/2, 1)$.

Theorem 1.[4] *Let A be of type (ω, M) with $\omega<\pi/2$. Then $-A$ is the infinitesimal generator of a semi-group $\{T_t\}_{t\geq 0}=\{\exp(-tA)\}$ with the following properties.*

a) *T_t has an analytic extension for $|\arg t|<\dfrac{\pi}{2}-\omega$.*

b) *In each smaller sector $|\arg t|<\dfrac{\pi}{2}-\omega-\varepsilon$, $\varepsilon>0$, T_t and $t\,dT_t/dt$*

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1) K. Yosida: Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them, Proc. Japan Acad., **36**, 86–89 (1960). For convenience we deviate from his notation in denoting by $-A$ instead of A the infinitesimal generator of a semi-group. The author is indebted to Professor Yosida for his suggestion to this problem.

2) A similar class of operators is considered by M. A. Krasnosel'skii and P. E. Sobolevskii, Doklady Acad. Nauk USSR, **129**, 499 (1959) and other Russian authors cited in this paper. But it appears that the semi-groups generated by $-A^{\alpha}$ are not considered by them.

3) This is a consequence of ii) if X is locally sequentially weakly compact, see T. Kato: Proc. Japan Acad., **35**, 467 (1959).

4) In case $M=1$, this theorem is contained in K. Yosida: Proc. Japan Acad., **34**, 337 (1958). Cf. also E. Hille and R. S. Phillips: Functional Analysis and Semi-groups, Am. Math. Soc. Colloq. Publ., Vol. 31, Theorems 12.8.1 and 17.5.1 (1957).