25. A Characterization of Real Analytic Functions

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1. Introduction. It is well known that a C^∞ function f is analytic on $[\alpha, \beta]$ if and only if there exist positive constants M and a such that

$$\sup_{x\in[\alpha,\beta]} |f^{(k)}(x)| \leqq Ma^k k!, \quad k=0,1,2,\cdots. \tag{1}$$

In this paper we prove a generalization of this fact for functions with several variables. Our main result is the following

Theorem. Let D be a domain in R^n, and let A be an elliptic differential operator of order m with constant coefficients. Then, for a function $f\in L^2_{\mathrm{loc}}(D)$ to be analytic in D it is necessary and sufficient that 1) for every k, $A^k f$ (in the sense of the distribution) belongs to $L^2_{\mathrm{loc}}(D)$, and that 2) for every compact $K\subset D$, there exist positive constants M and a such that*)

$$\| A^k f\|_K \leqq M(ak)^{mk}, \quad k=0,1,2,\cdots. \tag{2}$$

Recently E. Nelson gave a similar sufficient condition in the case where the coefficients of A are analytic [5]. His condition is essentially that

$$\| A^k f\|_K \leqq M(ak)^{k}, \quad k=0,1,2,\cdots. \tag{3}$$

It is highly desirable to obtain a result which includes the above two cases.

At the end of this paper an application will be given on the regularity of solutions of parabolic differential equations.

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2. Proof of the theorem. We prepare several lemmas. Lemma 1 can be proved by using Cauchy's integral formula and Taylor expansion.

Lemma 1. Let K be a compact convex set in R^n. A C^∞ function $f(x)$ defined on K is analytic if and only if there exist positive constants M and a satisfying

$$\sup_{x\in K} |D^p f(x)| \leqq Ma^{|p|} p!, \quad |p|=0,1,2,\cdots, \tag{4}$$

where

$$p=(p_1,\cdots,p_n), \quad |p|=p_1+\cdots+p_n,$$
$$D^p=\partial^{|p|}/\partial x_1^{p_1}\cdots\partial x_n^{p_n}, \quad p!=p_1!\,p_2!\cdots p_n!.$$

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*) $\|f\|_K=\left(\int_K |f(x)|^2 dx\right)^{1/2}$. But the theorem holds for norms other than the L^2-norm, too. For some system of differential operators an analogous theorem holds, of which Proposition 1 is a special case.