24. *Fractional Powers of Infinitesimal Generators and the Analyticity of the Semi-groups Generated by Them*

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1. Consider a one-parameter semi-group of bounded linear operators $T_t (t \geqq 0)$ on a Banach space X into X:

(1) $$T_t T_s = T_{t+s},\ T_0 = I \text{ (the identity operator)},$$

(2) $$\text{strong-}\lim_{t \to t_0} T_t x = T_{t_0} x,\ x \in X,$$

(3) $$\sup_t \| T_t \| < \infty.$$

The infinitesimal generator A of the semi-group T_t is defined by

(4) $$Ax = \text{strong-}\lim_{h \downarrow 0} h^{-1}(T_h - I)x.$$

It is known that A is a closed linear operator whose domain $D(A)$ is strongly dense in X. A fractional power

(5) $$-(-A)^\alpha,\ (0 < \alpha < 1),$$

of A was defined by S. Bochner[2)] and R. S. Phillips[3)] as the infinitesimal generator of the semi-group

(6) $$\hat{T}_t x = \hat{T}_{t,\alpha} x = \int_0^\infty T_\lambda x \, d\gamma_{t,\alpha}(\lambda),$$

where the measure $d\gamma_{t,\alpha}(\lambda) \geqq 0$ is defined through the Laplace integral

(7) $$\exp(-ta^\alpha) = \int_0^\infty \exp(-\lambda a) d\gamma_{t,\alpha}(\lambda),\ (t, a > 0 \text{ and } 0 < \alpha < 1).$$

The purpose of the present note is to prove that this semi-group $\hat{T}_t = \hat{T}_{t,\alpha}$ is analytic in t,[4)] or more precisely, that $\hat{T}_t$ belongs to the class of semi-groups introduced in a previous note.[5)]

For any $x \in X$ and for any $t > 0$, $\hat{T}_t x = \hat{T}_{t,\alpha} x$ is strongly differentiable in t, and $\hat{T}'_t x = \text{strong-}\lim_{h \downarrow 0} h^{-1}(\hat{T}_{t+h} - \hat{T}_t)x$ satisfies

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1) Dedicated to Prof. Zyoiti Suetuna on his 60th Birthday.

2) Diffusion equations and stochastic processes, Proc. Nat. Acad. Sci., **35**, 369-370 (1949).

3) On the generation of semi-groups of linear operators, Pacific J. Math., **2**, 343-369 (1952).

4) Originally the author proved the analyticity for the case $0 < \alpha \leqq 1/2$. It was communicated to Prof. Tosio Kato, and he has proved the analyticity for the case $0 < \alpha < 1$ by a more general approach. See the following paper by Prof. Kato. The author wishes to express his hearty thanks to Prof. Kato for the friendly discussion.

5) K. Yosida: On the differentiability of semi-groups of linear operators, Proc. Japan Acad., **34**, 337-340 (1958). Cf. E. Hille's class $H(\Phi_1, \Phi_2)$ of semi-groups in his book: Functional Analysis and Semi-groups, New York (1948).