53. A Characterization of Holomorphically Complete Spaces

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Given a connected complex space X, we denote by $A(X)$ the C-algebra of holomorphic functions on X. A C-homomorphism of $A(X)$ into C which preserves the constants is called a *character* of $A(X)$. Let X^* be the set of all characters of $A(X)$. The functions of $A(X)$ can be considered as functions on X^*. We shall consider X^* as a topological space: the open sets of X^* are those which can be represented as unions of sets of the form $f_1^{-1}(U_1)\frown\cdots\frown f_k^{-1}(U_k)$, where $f_1,\cdots,f_k$ are in $A(X)$, while $U_1,\cdots,U_k$ are open subsets of C ($f^{-1}(U)$ denotes the set of characters χ such that $\chi f\in U$). The space X^* is a Hausdorff space. We assign to each $x\in X$ a point $\theta(x)$ of X^* which is defined by $\theta(x)f=f(x)$ for every $f\in A(X)$. The mapping $\theta: X\to X^*$ is continuous.

Theorem. Let X be a connected complex space. Then X is holomorphically complete if and only if $\theta: X\to X^*$ is a homeomorphism.

For holomorphically complete spaces, see H. Cartan [1] and H. Grauert [2].

Proof. Suppose that X is holomorphically complete. Since X is holomorphically separable [2], the mapping θ is injective. Let χ be a point of X^*. We denote by M the maximal ideal Ker χ. Take $f_1\neq 0$ in M and decompose the analytic set $V^{(1)}=\{x\in X\mid f_1(x)=0\}$ of dimension $n-1$ (X being of dimension n) into irreducible components $V_i^{(1)}$. The family $(V_i^{(1)})$ being locally finite, we can find two points x_i, x_i' in $V_i^{(1)}$ for each i such that all the points are distinct and form an analytic set, of dimension 0, in X. By Theorem B on holomorphically complete spaces [1], we can find a function f in $A(X)$ such that $f(x_i)=0$ and $f(x_i')=1$ for every i. Let $f_2=f-\chi f$. Then $f_2\in M$ is not identically zero on each $V_i^{(1)}$. Decompose the analytic set $V^{(2)}=\{x\in X\mid f_1(x)=f_2(x)=0\}$ of dimension $n-2$ into irreducible components and find $f_3\in M$ as before. The repetition of such processes leads to the analytic set $V^{(n)}=\{x\in X\mid f_1(x)=\cdots=f_n(x)=0\}$ of dimension 0 in X, where $f_1,\cdots,f_n\in M$. Applying Theorem B again, we can find a function $f\in A(X)$ which takes different values at distinct points of $V^{(n)}$. Let $f_{n+1}=f-\chi f$. By Theorem A [1] we know that any finite subset of $A(X)$ without common zero generates $A(X)$ over itself. Therefore the functions $f_1,\cdots,f_{n+1}$ have at least one, and so only one, common zero, say x. For any $f\in M$, then functions $f_1,\cdots,f_{n+1},f$ have the common zero x and so $f(x)=0$, that is, $f\in\operatorname{Ker}\theta(x)$.