52. Characterizations of Spaces with Dual Spaces

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In the following we assume that spaces considered here are always completely regular and continuous functions are real-valued one. Let $X^*=\beta X-X$. We shall say that X has a dual space X^* if there is a homeomorphism of $\beta(X^*)$ onto βX which keeps X^* pointwisely fixed.[1] Then we write $X^{**}=(X^*)^*(=\beta(X^*)-X^*)=X$ or $\beta X=\beta(X^*)$. This notations may be justified by the properties A), B) and C) in §1. A subset B of X is said to be *inessential to* X if any *bounded* continuous function defined on $X-B$ is continuously extended over X. In §2 we shall show that if X has a dual space, then every compact subset of X is inessential to X and every finite subset of βX is inessential to βX. Using this results, we shall prove that X has a dual space if and only if every proper open subset of X whose complement is compact has a dual space.[2] We have given in [3] a stonean space with a dual space. In §3, we shall give examples of spaces with dual spaces among spaces of the following types: i) pseudo-compact spaces, ii) countably compact, Σ-product spaces, iii) countably compact, non-paracompact, normal spaces which have a uniform structure by the family of neighborhoods of the diagonal of product with itself, and iv) countably compact, non-normal spaces.

1. The proofs of the following properties are obvious.

A) Let Z and X be given spaces and let Y be a dense subset of Z. If two homeomorphisms φ and ψ from Z onto X coincide with each other on Y, then $\varphi(z)=\psi(z)$ for every $z\in Z$.

Let φ be a homeomorphism from $\beta(X^*)$ onto βX which keeps X^* pointwisely fixed.

B) If X has a dual space X^*, then every bounded continuous function f^* on X^* has a continuous extension $f=F\circ\varphi^{-1}$ over βX where F is a continuous extension of f^* over $\beta(X^*)$.

C) In B), let g be a bounded continuous function on X and g^* be

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1) The definition, in [3], of a dual space (the first row of p. 148 and the last row of p. 160) seems to be ambiguous, but the progression of arguments, in [3], with respect to a dual space was set in the sense of this paper.

2) This characterization may be of interest in view of the fact that the following conditions are equivalent for any X: i) X is a stonean space with a dual space, ii) any proper open subspace U of X has a dual space and $X-U$ is inessential to X, and iii) any proper dense subspace of X has a dual space. This fact is essentially proved in [3, Th. 12] (but with an inexact statement).