67. *Interior Regularity of Weak Solutions of the Time-Dependent Navier-Stokes Equation*

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§ 1. **Introduction.** It is an interesting problem of mathematical physics whether the time-dependent Navier-Stokes equation has a solution or not. To solve this problem, several authors proposed various weak solutions. In particular, E. Hopf[1] proved the existence, but not the uniqueness, of a weak solution which is global in time, whereas Kiselev and Ladyzenskaia[2] showed the local existence and uniqueness of a weak solution of a different type. In this note we show that the latter is actually a regular solution at least in the interior of the domain if the external force is smooth. We first sketch their result. The equation to be solved is

$$\partial u/\partial t-\Delta u+(u\nabla)u=-\nabla p+f,\quad \operatorname{div} u=0 \text{ in } D\subset E^3,$$
$$u|_{t=0}=a,\quad u|_{\partial D}=0 \ (\partial D \text{ is the boundary of } D).$$

Notations. A vector function belongs to C_0^∞ if its components are of class C_0^∞ (i.e. infinitely differentiable with compact support). $\overset{\circ}{K}_1(D)$ is a real Hilbert space obtained from $\overset{\circ}{K}(D)=\{f\,|\,f\in C_0^\infty(D),\ \operatorname{div} f=0\}$ by completion with the Dirichlet norm. $H_2(D)$ is a real Hilbert space consisting of all twice strongly differentiable vector functions with the norm $\left(\sum\int u_{i\ x_jx_k}^2dx+\sum\int u_{i\ x_m}^2dx+\sum\int u_n^2dx\right)^{1/2}$. $L^2(D)$ is a real Hilbert space of square integrable vector functions with the norm $||u||=(u,u)^{1/2}=\left(\sum\int u_i^2dx\right)^{1/2}$.

Assumptions. 1. D is a bounded domain in the three dimensional Euclidean space E^3. 2. The initial value a belongs to $H_2(D)\frown\overset{\circ}{K}_1(D)$. 3. The external force f and its time derivative $\partial f/\partial t$ belong to $L^2(D\times(0,l))$.

Conclusion. There exists a positive constant T such that in the domain $\Omega=D\times(0,T)$ a generalized solution $u(t)=u(x,t)$ exists uniquely with the following properties. 1. $u(t)\in\overset{\circ}{K}_1(D)$ for each t $(0<t<T)$; 2. u, ∇u, $\partial u/\partial t$, $\partial\nabla u/\partial t\in L^2(\Omega)$; 3. $u(t)$, $\nabla u(t)$, $\partial u(t)/\partial t\in L^2(D)$ for each t $(0<t<T)$ and their L^2 norms are bounded in t; 4. $u(t)\to a$ (strongly in $L^2(D)$ as $t\downarrow 0$); 5. For any sufficiently smooth solenoidal vector

1) E. Hopf: Math. Nachrichten, **4**, 213-231 (1950-1951).

2) A. A. Kiselev and O. A. Ladyzenskaia: Izv. Akad. Nauk SSSR, Seriya Mat., **21**, 655-680 (1957).