66. The Space of Bounded Solutions of the Equation $\Delta u = pu$ on a Riemann Surface

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Throughout this note we denote by R a Riemann surface. Suppose that p is a collection $\{p(z)\}$ of non-negative continuously differentiable functions $p(z)$ of local parameters $z=x+iy$ such that for any two members $p(z)$ and $p(z')$ in p there holds the relation

$$p(z')=p(z)|dz/dz'|^2.$$

We say that such a p is a *density* on R. We consider the partial differential equation of elliptic type

$$\Delta u(z)=p(z)u(z), \tag{1}$$

which is invariantly defined on R. We denote by $B_p(R)$ the totality of real-valued bounded solutions of this equation (1) on R. Here a solution of (1) is always assumed to be twice continuously differentiable. Then $B_p(R)$ is a Banach space with the uniform norm

$$||u||=\sup_R |u|.$$

We are interested in the comparison problem of Banach space structures of $B_p(R)$ for different choices of densities p. It is remarked, as Ozawa proved in [3], that if R is of parabolic type, then $B_0(R)$ is the real number field and $B_p(R)$ consists of only zero unless $p\equiv 0$. Hence we may exclude this trivial case as far as we are concerned with spaces $B_p(R)$. So we assume that R is of hyperbolic type throughout this note unless the contrary is stated. Concerning this comparison problem Royden [4] proved that if there exists a positive constant a such that

$$a^{-1}p\leq q\leq ap$$

holds on R except a compact subset of R, then Banach spaces B_p and B_q are isomorphic. In this note we give a different criterion for B_p and B_q to be isomorphic and state an application of this to removable singularities of bounded solutions of (1).

Theorem 1. *If two densities p and q on R satisfy the condition*

$$\iint_R |p(z)-q(z)|\,dxdy<\infty, \tag{2}$$

then Banach spaces $B_p(R)$ and $B_q(R)$ are isomorphic.

Proof.[1] Let $\{R_n\}$ be an exhaustion of R, i.e. R_n is a subdomain of R whose closure is compact and whose relative boundary ∂R_n consists of a finite number of closed analytic Jordan curves and moreover

1) For elementary knowledge concerning the equation $\Delta u=pu$ on a Riemann surface, refer to Myrberg [1, 2] and also to Royden [4, section 1].