64. A Galois Theory for Finite Factors

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According to a closed analogy between theories of classical simple algebras and continuous finite factors, it is natural to ask that a continuous finite factor obeys a kind of Galois theory. Literally, it is known that I. M. Singer [4] gave an attempt in this direction.

This note will present a trial towards it in the following

THEOREM. *If A is a continuous finite factor acting standardly on a separable Hilbert space H, if G is a finite group of outer automorphisms of A, if B is the subfactor of A consisting of all elements invariant under G, and if moreover the commutor B' of B is finite. Then, the lattices of all subgroups of G and of all intermediate subfactors between B to A are dually isomorphic under the Galois correspondence which carries a subgroup F to an intermediate subfactor C invariant under F in element-wise.*

It is expected that the assumption on B' is provable from the finiteness of G for which the authors hope to discuss in the next occasion. It is also to be remarked that the continuity assumption on A in the theorem is superfluous since a discrete finite factor has no non-trivial group of outer automorphisms.

1. Since A acts standardly on H, there is a unitary u_g for any g such as

$$x^g = u_g x u_g^*, \tag{1}$$

where x^g means the action of g on $x \in A$. Throughout the remainder, for the sake of convenience, it is to be assumed that the correspondence $g \to u_g$ satisfies

$$u_{g^{-1}} = u_g^*. \tag{2}$$

It is to be noticed that u_g belongs to B', since $x = x^g = u_g x u_g^*$ by the assumption.

LEMMA 1. *By* (1), *g gives an outer automorphism on A'.*

If $x \in A'$, then for any $a \in A$, (1) and (2) imply

$$ax^g = au_g x u_g^* = u_g a^{g^{-1}} x u_g^* = u_g x a^{g^{-1}} u_g^* = u_g x u_g^* a = x^g a,$$

which shows that g conserves A'. Hence (1) gives an automorphism on A'. If it is inner, then there is a unitary $w \in A'$ such that $x^g = w^* x w$ or $u_g x u_g^* = w^* x w$ for any $x \in A'$, whence $wu_g x = xwu_g$ for any $x \in A'$, that is, wu_g commutes with every element of A'. Hence the unitary operator $w' = wu_g$ belongs to A. Therefore, by $w \in A'$,

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