80. Some Applications of the Maximum Principle for Subharmonic Functions

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Let F be hyperbolic Riemann surface and p_0 be a point fixed on F. Let $g(p, p_0)$ be the Green function of F with the pole at p_0 and $h(p, p_0)$ be conjugate to it. G_r is the domain such that $g(p, p_0) > -\log r$ with the boundary C_r. For the points $\tilde{p}, \tilde{p}_0$ on $\tilde{F}$, we define $\tilde{g}(\tilde{p}, \tilde{p}_0)$, $\tilde{h}(\tilde{p}, \tilde{p}_0)$ similarly.

We define the *modulus* of $p, \tilde{p}$ by the relation

$$|p|_F = e^{-g(p,p_0)}, \quad |\tilde{p}|_{\tilde{F}} = e^{-\tilde{g}(p,\tilde{p}_0)},$$

respectively. The ordinary modulus is denoted by '$|\ \ |$'.

1. Let f be an analytic mapping of F into $\tilde{F}$. Then $\tilde{g}(f(p), \tilde{p}_0)$ is harmonic except for the points at which $f(p) = \tilde{p}_0$, and for such points $\tilde{g}(f(p), \tilde{p}_0) = \infty$. Therefore, $\log|f(p)|_{\tilde{F}}$ is subharmonic on F.

Theorem 1 (Schwarz). $|f(p)|_{\tilde{F}} \leqq |p|_F$ *for* $p \in F$.

Proof. Consider the function

$$u(p) = \log|f(p)|_{\tilde{F}} + g(p, p_0).$$

Since $\log|f(p)|_{\tilde{F}}$ is subharmonic and $g(p, p_0)$ is harmonic on $F' = F - p_0$, $u(p)$ is subharmonic on F'. Let z be a local parameter in the neighborhood V of p_0. The function $w(p) = \exp\{-\tilde{g}(f(z), \tilde{p}_0) - i\tilde{h}(f(z), \tilde{p}_0)\}$ is analytic in z. Since we have in V

$$u(z) = -\log|w(z)/z| + u_1(z), \quad u_1 \text{ is harmonic in } V,$$

and $w(0) = 0$, $u(z)$ is subharmonic in V. Thus $u(p)$ is subharmonic on F.

For an arbitrary $r < 1$, $u(p) \leqq \log r$ on C_r. From the maximum principle we obtain the same inequality in G_r. As $r \to 1$, we have $u(p) \leqq 0$ on F, and this proves the theorem.

Corollary 1. *If* $f(p)$ *is an analytic function on* F *such that* $|f(p)| \leqq M$ *and* $f(p_0) = 0$, *then* $|f(p)| \leqq M|p|_F$.

This is easily seen by taking the plane domain $|w| \leqq M$ as $\tilde{F}$ in the theorem.

Theorem 2. *Let* $p_1, p_2, \cdots, p_n$ *be the points such that* $f(p_i) = \tilde{p}_0$, $i = 1, 2, \cdots, n$, *then*

$$|f(p_0)|_{\tilde{F}} \leqq \prod_{i=1}^{n} |p_i|_F.$$

Proof. We assume that $f(p_0) \neq \tilde{p}_0$, otherwise the theorem is trivial. The function $u(p) = \log|f(p)|_{\tilde{F}} + \sum_{i=1}^{n} g(p, p_i)$ is subharmonic on F.