79. *On a Metric Characterizing Dimension*

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As well known, a separable metric space has dimension$\leqq n$ if and only if it admits a topology-preserving metric such that almost all of the spherical nbds (=neighborhoods) of any point have boundaries of dimension$\leqq n-1$ [1, 2]. To extend this theorem to a non-separable metric space and its covering dimension, we must face two problems. The first of them is how to modify the above condition of metric to fit it for the non-separable case, because in that case, we can not regard this condition as a sufficient condition for n-dimensionality so far as the well-known conjecture, $\dim R = \operatorname{ind} \dim R$, has not yet been solved. The second is how to manage the proof in a non-separable metric space R without a measure, because, although the above theorem was originally proved by virtue of Szpilrajn's theorem on the so-called p-dimensional measure and dimension [3], the measure does not work at all in a general metric space.

After all we can insist the following theorem for a general metric space R and the covering dimension of R.

Theorem. *A metric space R has dimension$\leqq n$ if and only if it admits a topology-preserving metric such that the spherical nbds $S_{\frac{1}{2^i}}(p)$, $i=1,2,\cdots$ of any point p have boundaries of dimension$\leqq n-1$ and such that $\{S_{\frac{1}{2^i}}(p) \mid p \in R\}$ is closure preserving for every i.*

Remark. We denote by $S_\varepsilon(p)$ the spherical nbd of p with a radius ε, i.e. $S_\varepsilon(p)=\{q \mid \rho(p,q)<\varepsilon\}$. We call a collection $\{S_\gamma \mid \gamma \in \Gamma\}$ of subsets '*closure preserving*' if $\bigcup_{\gamma \in \Delta} \bar{S}_\gamma = \overline{\bigcup_{\gamma \in \Delta} S_\gamma}$ for any subset Δ of Γ. The metric in this theorem is a particular one; the metrics of Euclidean spaces, for instance, do not satisfy the second condition. To replace $S_{\frac{1}{2^i}}(p)$, $i=1,2,\cdots$ in this theorem by more spherical nbds will be another interesting problem.

Proof. *Sufficiency*: First, let us note that $\{BS_{\frac{1}{2^i}}(p) \mid p \in A\}$ is closure preserving and $B[\bigcup\{S_{\frac{1}{2^i}}(p) \mid p \in A\}] \subset \bigcup\{BS_{\frac{1}{2^i}}(p) \mid p \in A\}$ for every subset A of R, for $\{S_{\frac{1}{2^i}}(p) \mid p \in R\}$ is closure preserving, where we denote by BS the boundary of S. Hence $\dim \bigcup\{BS_{\frac{1}{2^i}}(p) \mid p \in A\} \leqq n-1$ follows from $\dim BS_{\frac{1}{2^i}}(p) \leqq n-1$, $p \in A$ by virtue of a theorem due to K.