77. On the Uniform Approximation by Meromorphic Functions on a Riemann Surface

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The Weierstrass' basic theorem on uniform approximation to continuous function over an interval was first generalized by J. L. Walsh in the complex plane.[1] *Let J be a Jordan curve of the finite z-plane containing in its interior the origin. Then an arbitrary function continuous on J can be uniformly approximated on J by a polynomial in z and $1/z$.*

Let J be a Jordan arc of the finite z-plane. Then an arbitrary function continuous on J can be uniformly approximated on J by a polynomial in z.

A chance to generalize these theorems on an arbitray Riemann surface was given in 1948. H. Behnke and K. Stein discussed the Runge's theorem on a non-compact Riemann surface.[2] By their approach to Runge's theorem we can easily verify the cited theorems on a non-compact Riemann surface.[3]

Let R be an arbitrary non-compact Riemann surface and J be a closed Jordan curve on R. Then an arbitrary function $f(p)$ continuous on J can be uniformly approximated on J by a function meromorphic on R with poles one and at least one in each one of the component of the complement to J. In particular if J does not separate R, $f(p)$ can be approximated uniformly on J by a function holomorphic in R.

In fact, there exists (1) an annular region G of R containing J, (2) an annulus $a\leqq|z|\leqq b$ in the complex plane, (3) a one-one conformal mapping τ of G onto the annulus such that the boundary components of ∂G are mapped on $|z|=a$, $|z|=b$ respectively. Let $J^*=\tau(J)$. Then $f(\tau^{-1}(z))$ is continuous on J^*. Since $f(\tau^{-1}(z))$ can be approximated uniformly on J^* by a polynomial in z and $1/z$, we can find a rational function $R(z)$ such that for every $\varepsilon>0$ $|f(\tau^{-1}(z))-R(z)|<\frac{\varepsilon}{2}$ holds on J^*. Further since $R(\tau(p))$ is holomorphic on $\overline{G}$, we can find a mero-

1) J. L. Walsh: Interpolation and Approximation by Rational Function in the Complex Domain, Amer. Math. Colloq. Pub., 36-48 (1936).

2) H. Behnke and K. Stein: Entwicklungen analytischer Funktionen auf Riemannschen Flächen, Math. Ann., **120**, 430-461 (1948).

3) This is not the case for functions of many variables. Cf. J. Wermer: Polynomial approximation on an arc in C^3, Ann. of Math., **62**, no. 2 (1955).