75. A Note on the Milnor's Invariant λ' for a Homotopy 3-sphere

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1. Let M be a differentiable $(4k-1)$-manifold which is a homology sphere and the boundary of some parallelizable manifold W. (The word "manifold" will mean a "compact" manifold throughout in this note.) The intersection number of two homology classes α, β of W will be denoted by $\langle\alpha, \beta\rangle$. Let $I(W)$ be the index of the quadratic form

$$\alpha \rightarrow \langle\alpha, \alpha\rangle,$$

where α varies over the Betti group $H_{2k}(W)/(\text{torsion})$. Integer coefficients are to be understood.

Define I_k as the greatest common divisor of $I(M)$ where M ranges over all almost parallelizable manifolds[1] without boundary of dimension $4k$. The residue class $\frac{1}{8}I(W)$[2] modulo $\frac{1}{8}I_k$ will be denoted by $\lambda'(M)$.

Then J. Milnor [1] showed the followings:

(1) $\lambda'(M)$ depends only on the J-equivalence[3] class of M,

(2) λ' gives rise to an isomorphism onto

$$\Lambda' : \Theta^{4k-1}(\partial\pi) \rightarrow Z_{\frac{1}{8}I_k} \quad \text{provided that } k>1,$$

where $\Theta^{4k-1}(\partial\pi)$[4] is the set of all J-equivalence classes of homotopy $(4k-1)$-spheres which are the boundaries of some parallelizable manifolds.

Finally, in the list of unsolved problems (see [1]), he proposed the following:

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1) A manifold M will be called almost parallelizable if there exists a finite subset F so that $M-F$ is parallelizable.

2) The index $I(W)$ of an almost parallelizable manifold is always divisible by 8, provided that ∂W is a homology sphere (see J. Milnor [1]).

3) Two unbounded manifolds M_1, M_2 of the same dimension are J-equivalent if there exists a manifold W such that

(1) the boundary ∂W is the disjoint union of M_1 and $-M_2$,

and

(2) both M_1 and M_2 are deformation retracts of W.

4) $\Theta^{4k-1}(\partial\pi)$ forms an abelian group under the sum operation $\#$, where $\#$ means the following. Let M_1, M_2 be connected differentiable (or combinatorial) manifolds of the same dimension n. The differentiable (or combinatorial) sum $M_1 \# M_2$ is obtained by removing a differentiable (or a combinatorial) n-cell from each, and then pasting properly the resulting boundary together (see J. Milnor [1, §2] and H. Seifert-W. Threlfall [7, Problem 3, p. 218]).