74. On the Theory of Non-linear Operators

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(Comm. by K. KUNUGI, M.J.A., June 13, 1960)

In this note, we consider the eigenvalue problem of some kinds of non-linear integral operators of Hammerstein type. In §1, we will give a general principle, and, in §2, we will apply it to the case of integral operators of Hammerstein type defined on Banach function spaces.

§1. Let R be a Banach space[1] and $\overline{R}$ be its conjugate space. For $\phi \in R$ and $\Phi \in \overline{R}$, we denote the value of Φ at ϕ by (Φ, ϕ).

A functional (in general, non-linear) $F(\phi)$ $(\phi \in R)$ is said to be *Fréchet-differentiable at* ϕ_0 if there exists an operator $\operatorname{grad} F = \mathfrak{f} \in (R \rightarrow \overline{R})$ such that

$$F(\phi_0+\phi)-F(\phi_0)=(\mathfrak{f}\phi_0, \phi)+r(\phi_0, \phi),$$

$$\lim_{\|\phi\|\rightarrow 0} \frac{|r(\phi_0, \phi)|}{\|\phi\|}=0.$$

$F(\phi)$ is said to be *increasing*[2] (*decreasing*) if

$$\lim_{\|\phi\|\rightarrow\infty} F(\phi)=+\infty \quad (-\infty).$$

A linear operator $K \in (R \rightarrow \overline{R})$[3] is said to be
symmetric if $(K\phi, \psi)=(K\psi, \phi)$ $(\phi, \psi \in R)$;
positive definite if $(K\phi, \phi) \geqq 0$ $(\phi \in R)$.[4]

Theorem 1. *Let R be a reflexive Banach space and $F(\phi)$ be convex, increasing, $F(0)=0$ and Fréchet-differentiable at any point of R. Let K be completely continuous, symmetric and positive definite. Then, for any number $\rho>0$ there exists a number $\lambda_\rho>0$ and an element $\phi_\rho \neq 0$ such that*

$$K\phi_\rho=\lambda_\rho \mathfrak{f}\phi_\rho, \quad F(\phi_\rho)=\rho.$$

Proof. For any $\rho>0$, put

$$V_\rho=\{\phi \in R: F(\phi) \leqq \rho\}.$$

Then, V_ρ is weakly closed and bounded. In fact, by virtue of continuity and convexity of $F(\phi)$, V_ρ is convex and closed, and the assumption that $F(\phi)$ is increasing implies that V_ρ is bounded.

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1) In this note, we consider only *real* Banach spaces.

2) This definition is due to [2, p. 302].

3) By $(R_1 \rightarrow R_2)$, we denote the set of all operators whose domains are R_1 and ranges are in R_2.

4) For these properties of K, see [6], where a kind of eigenvalue problem of such K has been considered.