100. General Crossnorms

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In the present note we shall generalize the construction of "general" crossnorm given by R. Schatten in Appendix II in his book: A theory of cross-spaces, Ann. of Math. Studies, no. 26, Princeton (1950) to which we shall refer as [TCS] in this note. And we shall prove some properties of our general crossnorms.

Our construction of general crossnorms is a slight modification of R. Schatten's method. Consequently our proofs of Lemmas 1, 2, 3 and Theorem 1 are almost analogous to those of his Theorems, but for the benefit of readers we shall prove them. Moreover we shall make use of notations, terminologies and results shown in [TCS] without reservation.

Throughout the present note we shall assume that $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ represent perfectly general Banach spaces while $\boldsymbol{B}_1^*$ and $\boldsymbol{B}_2^*$ stand for their conjugate spaces respectively.

LEMMA 1. *If p, q are positive and $\frac{1}{p}+\frac{1}{q}=1$, then for positive numbers a, b, we have*

$$\frac{1}{\frac{a}{p}+\frac{b}{q}}\leqq\frac{1}{pa}+\frac{1}{qb}.$$

Proof. Immediately.

LEMMA 2. *If p, q are positive and $\frac{1}{p}+\frac{1}{q}=1$, then for any two norms α, β, we have*

$$\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)'\leqq\frac{\alpha'}{p}+\frac{\beta'}{q}.$$

Proof. Let $\widetilde{F}\in\boldsymbol{B}_1^*\odot\boldsymbol{B}_2^*$ be fixed. By Lemma 1, for any non-zero $\widetilde{f}\in\boldsymbol{B}_1\odot\boldsymbol{B}_2$ we have

$$\frac{|\widetilde{F}(\widetilde{f})|}{\frac{\alpha(\widetilde{f})}{p}+\frac{\beta(\widetilde{f})}{q}}\leqq\frac{\widetilde{F}(\widetilde{f})}{p\alpha(\widetilde{f})}+\frac{\widetilde{F}(\widetilde{f})}{q\beta(\widetilde{f})}.$$

Thus, definition of "associated" norm furnishes the proof.

We proceed with our construction:

Put $\alpha_{p,1}=\frac{\gamma}{p}+\frac{\gamma'}{q}$, $\alpha_{q,1}=\frac{\gamma}{q}+\frac{\gamma'}{p}$ and $\alpha_{p,n}=\frac{\alpha_{p,n-1}}{p}+\frac{\alpha'_{q,n-1}}{q}$,