99. A Note on Subdirect Decompositions of Idempotent Semigroups

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A subsemigroup B of the direct product $B_1\times B_2\times\cdots\times B_n$ of bands (i.e. idempotent semigroups) $B_1, B_2,\cdots, B_n$ is called a *subdirect product* of $B_1, B_2,\cdots, B_n$ if every i,

$$\xi_i(B)=B_i$$

where ξ_i is the i-th projection of $B_1\times B_2\times\cdots\times B_n$.

Let $\mathfrak{R}_1, \mathfrak{R}_2,\cdots, \mathfrak{R}_m$ be congruences on a band S. Then the set $S^*=\{(\varphi_1(a), \varphi_2(a),\cdots, \varphi_m(a)) : a\in S\}$, where each φ_i is the natural homomorphism of S to $S/\mathfrak{R}_i$, becomes a subdirect product of $S/\mathfrak{R}_1, S/\mathfrak{R}_2,\cdots, S/\mathfrak{R}_m$. Such S^* is called the *natural representation* of S induced by $\mathfrak{R}_1, \mathfrak{R}_2,\cdots, \mathfrak{R}_m$, and denoted by $S/\mathfrak{R}_1\circ S/\mathfrak{R}_2\circ\cdots\circ S/\mathfrak{R}_m$. Especially, it has been shown by Birkhoff [1] that if $\mathfrak{R}_1\cap\mathfrak{R}_2\cap\cdots\cap\mathfrak{R}_m=0$,[1] then $S/\mathfrak{R}_1\circ S/\mathfrak{R}_2\circ\cdots\circ S/\mathfrak{R}_m$ is an isomorphic representation of S.

Another important type of subdirect product, which is often used in the study of bands, is *spined product* introduced by Kimura [2]:

Let $S_1, S_2,\cdots, S_n$ be bands having Γ as their structure semilattices. And let $\mathfrak{D}_i : S_i\sim\Sigma\{S_i^\gamma : \gamma\in\Gamma\}$, for each i with $1\leqq i\leqq n$, be the structure decomposition of S_i.[2] Then, the set $S=\bigcup\{S_1^\gamma\times S_2^\gamma\times\cdots\times S_n^\gamma : \gamma\in\Gamma\}$ becomes a subdirect product of $S_1, S_2,\cdots, S_n$. Such S is called the spined product of $S_1, S_2,\cdots, S_n$ with respect to Γ, and denoted by $S_1\bowtie S_2\bowtie\cdots\bowtie S_n\,(\Gamma)$.

The main purpose of this paper is to present the following representation theorem which clarifies the relation between such two special kinds of subdirect product.

Theorem. *Let S be a band, and $\mathfrak{D} : S\sim\Sigma\{S_\gamma : \gamma\in\Gamma\}$ its structure decomposition. Let $\mathfrak{R}_1, \mathfrak{R}_2,\cdots, \mathfrak{R}_n$, $n\geqq 2$, be congruences on S.*

If $\mathfrak{R}_1, \mathfrak{R}_2,\cdots, \mathfrak{R}_n$ satisfy

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1) The ordering in the set Ω of all congruences on S is as follows: For $\mathfrak{A}, \mathfrak{B}\in\Omega$, $\mathfrak{A}\leqq\mathfrak{B}$ if and only if for $x, y\in S$ $x\mathfrak{A}y$ implies $x\mathfrak{B}y$. The element 0 will denote the least element of Ω in the sense of this ordering.

2) Let S be a band. Then, there exist a semilattice Γ and a disjoint family of rectangular subsemigroups of S indexed by Γ, $\{S_\gamma : \gamma\in\Gamma\}$, such that

$$S=\cup\{S_\gamma : \gamma\in\Gamma\}$$

$$\text{and}\qquad S_\alpha S_\beta\subset S_{\alpha\beta}\qquad\text{for } \alpha, \beta\in\Gamma$$

(see McLean [3]). In this case Γ is determined uniquely up to isomorphism, and called the structure semilattice of S. Further this decomposition, say $\mathfrak{D}$, gives a congruence called the structure decomposition of S and denoted by $S\sim\Sigma\{S_\gamma : \gamma\in\Gamma\}$.