96. A Necessary and Sufficient Condition under which $\dim(X\times Y)=\dim X+\dim Y$

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§ 1. **Introduction.** Let X and Y be locally compact fully normal spaces. It is well known that the relation $\dim(X\times Y)\leqq\dim X+\dim Y$ holds, where dim means the covering dimension (cf. [12]). But, the following stronger relation $(*)$ does not hold in general:

$$(*)\qquad \dim(X\times Y)=\dim X+\dim Y.$$

Some necessary conditions in order that the relation $(*)$ hold have been obtained by E. Dyer[1] and the author.[2] However, these conditions are not a sufficient condition.[3] The object of this paper is to obtain a necessary and sufficient condition under which the relation $(*)$ is true.

Let G be an abelian group. The *homological dimension of X with respect to G* (notation: $D_*(X:G)$) is the largest integer n such that there exists a pair (A, B) of closed subsets of X whose n-dimensional (unrestricted) Čech homology group $H_n(A, B:G)$[4] with coefficients in G is not zero. A space X is called *full-dimensional with respect to G* if $D_*(X:G)=\dim X$. Let us use the following notations: R=the additive group of all rationals, Z=the additive group of all integers, R_1=the factor group R/Z, Q_p=the p-primary component of R_1 for a prime p, Z_q=the cyclic group with order $q(=Z/qZ)$, $Z(\mathfrak{a}_p)$=the limit group of the inverse system $\{Z_{p^i}:h_i^{i+1};\ i=1,2,\cdots\}$, where h_i^{i+1} is a natural homomorphism from $Z_{p^{i+1}}$ onto Z_{p^i}. We shall prove the following theorem.

Theorem. *Let X and Y be locally compact fully normal spaces. In order that the relation* $\dim(X\times Y)=\dim X+\dim Y$ *hold it is necessary and sufficient that at least one of the following four conditions be satisfied:*

(1) *X and Y are full-dimensional with respect to R.*

(2) *X and Y are full-dimensional with respect to Z_p for a prime p.*

(3) *X and Y are full-dimensional with respect to $Z(\mathfrak{a}_p)$ and Q_p for a prime p respectively.*

(4) *X and Y are full-dimensional with respect to Q_p and $Z(\mathfrak{a}_p)$ for a prime p respectively.*

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1) Cf. [5, Theorem 4.1].
2) Cf. [10, Theorem 5].
3) Cf. [5, p. 141].
4) Cf. [4] and [9, p. 96].
5) Cf. [8, p. 385].