93. A Theorem on Flat Couples

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In this short note, I will prove a theorem in homological algebra and its corollary, which is well known in ideal theory in integral domains.

Throughout this note any ring is assumed to be commutative and have a unit element which acts as the identity operator on any module over the ring. We will call the pair (R, R') of a ring R and its overring R' a flat couple, if R'/R is flat as an R-module [7]. A ring R is called semi-hereditary if every finitely generated ideal of R is R-projective [1]. Then we have the

THEOREM. *Let R be a semi-hereditary ring and R' be an integral (or module finite) extension ring of R. Then, (R, R') is a flat couple.*

The theorem is obtained directly from the following two lemmas.

LEMMA 1. *A semi-hereditary ring is integrally closed in its full ring of quotients.*

PROOF. Let R be a semi-hereditary ring and K be its full ring of quotients. Let x be an element of K and be integral over R and

$$x^n+r_1x^{n-1}+\cdots+r_n=0$$

be an equation of integral dependence satisfied by x over R. There exists a non-zerodivisor r of R such that $rx^{n-i}\in R$ for $i=0,1,\cdots,n-1$. Since $x^{n+1}=-(r_1x^n+\cdots+r_nx)$, rx^{n+1} is also in R. Thus we have $rx^i\in R$ for $i=1,2,\cdots$. Now, we consider an ideal I of R generated by $(rx^i;\ i=1,2,\cdots)$. Since this ideal I is finitely generated (in fact, generated by $rx, rx^2,\cdots, rx^n$) and R is semi-hereditary, I is projective and by Cartan-Eilenberg [1, VII, 3.1] there exist R-homomorphisms $\varphi_i: I\to R$ such that $y=\sum_{i=1}^{n}\varphi_i(y)rx^i$ for all $y\in I$. Thus since $rx\in I$, it follows

$$rx=\sum_{i=1}^{n}\varphi_i(rx)rx^i=\sum_{i=1}^{n}\varphi_i(r^2x^{i+1})=\sum_{i=1}^{n}\varphi_i(rx^{i+1})r,$$

and since r is a non-zerodivisor, we have $x=\sum_{i=1}^{n}\varphi_i(rx^{i+1})\in R$. This shows that R is integrally closed in K.

Let A be an R-module and a be a non-zero element of A. We say that a is an R-torsion element if $ra=0$ for some non-zerodivisor r of R, and A is called R-torsion-free if A has no R-torsion element except zero.

LEMMA 2. *A ring R is integrally closed in its full ring of*