118. Characterizations of Spaces with Dual Spaces. II

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1960)

We have proved in [1] that, for a completely regular space X, the following conditions are equivalent: i) X is a stonean space with a dual space, ii) any proper open subspace U of X has a dual space and $X-U$ is inessential to X *) and iii) any proper dense subspace of X has a dual space. For a completely regular space with a dual space, its characterizations are given in [2]. For instance, we have proved that X has a dual space if and only if any proper open subspace U of X whose complement is compact has a dual space. This is a generalization of iii) mentioned above because the subspace U whose complement is compact is dense in X if X has a dual space.

In this paper, we shall first generalize ii) mentioned above, that is, we shall show that X has a dual space if and only if any proper open subspace U of X has a dual space. But this does not mean that a dual space of U is $\overline{U}(\text{in } \beta X)-U$. If U has always $\overline{U}(\text{in } \beta X)-U$ as a dual space, then X becomes a stonean space with a dual space. Next, since examples of spaces with dual spaces given in [2] are all pseudo-compact, we shall give here non-pseudo-compact spaces with non-pseudo-compact dual spaces.

1. We shall first state a useful lemma.

Lemma 1. *Let F be a closed subset of X, if f is a bounded continuous function on $X-F$, then f has a continuous extension over $\beta X-\overline{F}(\text{in } \beta X)$.*

A proof of this lemma follows from the proof of ii) of Lemma 1 in [2].

Theorem 1. *X has a dual space if and only if any proper open subspace has a dual space.*

Proof. If any open subspace has a dual space, then X has a dual space by i)$\leftrightarrow$ii) of Theorem 2 in [2]. Thus, to prove the theorem, it is sufficient to show the converse. Suppose that X has a dual space. If U is open X, any point of U has no compact neighborhoods and any bounded continuous function on U can be continuously extended over M where $M=X^*-E$, $E=\overline{(X-U)}(\text{in } \beta X)$ by Lemma 1. It is easily seen that $E\frown X=X-U$ and M is an open subspace of X^* whose points have no compact neighborhoods. Now we consider the Stone-

*) See footnote 2) in [2].