117. On the Maximum Principles of Second Order Elliptic Differential Equations

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1960)

The aim of this note is to extend the well-known maximum principle of E. Hopf[1] concerning the general second order elliptic differential equation

(1) $$F(x, u, u_k, u_{ij})=0,$$ [2]

where $u_i=\partial u/\partial x_i$, $u_{ij}=\partial^2 u/\partial x_i \partial x_j$.

In this note we shall derive two kinds of the maximum principles under the following

Assumptions. I. The function $F(x, u, p_k, r_{ij})$ is defined in the domain $\mathfrak{D}: x\in G, |u|, |p_k|, |r_{ij}|<\infty$, where G is any domain in the Euclidean n-space.

II. $F(x, u, p_k, r_{ij})$ is continuously differentiable with respect to the arguments r_{ij} provided that the other arguments x, u, p_k remain fixed. Moreover, for every compact subset $\mathfrak{A}$ of $\mathfrak{D}$ there exists a constant $A>0$ such that

$$A^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \xi_i \xi_j \leqq A|\xi|^2$$

for any $(x, u, p_i, r_{ij})\in\mathfrak{A}$, and for any n-tuple $\xi=(\xi_1,\cdots,\xi_n)$.

III. $F(x, u, p_k, r_{ij})$ satisfies the Lipschitz condition with respect to the arguments u, p_i, r_{ij} in every compact subset of the domain $\mathfrak{D}$.

THEOREM I. *Let $u^{(1)}(x)$ and $u^{(2)}(x)$ be two $C^2(G)$-functions which satisfy the differential inequalities*

(2) $$F(x, u^{(1)}, u_k^{(1)}, u_{ij}^{(1)})\leqq 0$$

and

(3) $$F(x, u^{(2)}, u_k^{(2)}, u_{ij}^{(2)})\geqq 0$$

in the domain G respectively. We assume further that $u^{(2)}(x)\leqq u^{(1)}(x)$ in the domain G. Then we have the following alternative:

Either $u^{(2)}(x)\equiv u^{(1)}(x)$ *in the domain G,*

or $u^{(2)}(x)< u^{(1)}(x)$ *throughout in G.*

Proof. The proof will be carried out by reducing the theorem to the less general lemma.

LEMMA. *If the function F is of the form*

1) E. Hopf: Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, Sitzungsberichte Preuss. Akad. Wiss., **19**, 147-152 (1927).

2) We denote by x the point $(x_1,\cdots,x_n)$ of the Euclidean n-space.