115. On Generalized Peano's Theorem concerning the Dirichlet Problem for Semi-linear Elliptic Differential Equations

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The purpose of this note is to prove a theorem which concerns the Dirichlet problem for semi-linear elliptic differential equations and is similar to Peano's theorem concerning the initial value problem of ordinary differential equations of the first order.[1] The precise statement of the theorem will be given in §2.

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1. **Preliminaries.** In this note we shall consider the semi-linear elliptic differential equation

$$(1) \qquad \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u)\ ^{2)}$$

in a bounded domain G under the following assumptions.

Assumptions. 1°. G is a bounded Poincaré domain in the Euclidean m-space; i.e. for each boundary point x of G there exist one half C_x of a circular cone with vertex x and a closed sphere K_x with center x such that

$$C_x \frown K_x \frown \overline{G} = \{x\}.^{3)}$$

2°. The symmetric matrix $\|a_{ij}(x)\|$ is continuous and positive-definite in the closure $\overline{G}$ of G.

3°. The function $f(x, u, p)$ $(p=(p_1, \cdots, p_m))$ is defined in $\mathfrak{D}: x \in \overline{G}$, $|u| < \infty$, $|p| < \infty$ and Hölder-continuous (with some exponent α, $0 < \alpha < 1$) in every compact subset of $\mathfrak{D}$. Further $f(x, u, p)$ is non-decreasing with respect to u; i.e.

$$f(x, u, p) \leqq f(x, \overline{u}, p) \quad \text{provided } x \in \overline{G},\ u < \overline{u},\ |p| < \infty.$$

Moreover, we assume that for every constant $M > 0$ there exist two constants $B(M)$ and $F(M)$ such that

$$|f(x, u, p)| \leqq B(M)|p| + F(M)$$

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1) As for generalized Peano's theorem concerning the Dirichlet problem see T. Satō: Sur l'equation aux derivées partielles $\Delta z = f(x, y, z, p, q)$ I, Compositio Math., **12**, 157–177 (1954); II, ibid., **14**, 152–172 (1959). See, in particular, Théorème 3 of the second note.

2) Here $x = (x_1, \cdots, x_m)$ and grad $u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_m)$.

3) We denote by $\overline{G}$ the closure $G + \Gamma$ of the domain G, where Γ is the boundary of G.