114. On the Cosine Problem

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1960)

1. **Introduction.** The main object of the present note is to establish the following theorem, which will answer in the affirmative to the cosine problem proposed by S. Chowla in connexion with a question concerning zeta functions (cf. [1]):

Theorem 1. *Let K be an arbitrary positive number. Then there exists a natural number $n_0=n_0(K)$ such that for any $n\geqq n_0$ and any set of n distinct positive integers $m_1, m_2,\cdots, m_n$ we have*

$$\min_{0\leqq x<2\pi}(\cos m_1x+\cos m_2x+\cdots+\cos m_nx)<-K.$$

Here we may take

(1) $$n_0(K)=\max(2^{48}, [8K^2]^{8[256K^4]}),$$

which is, of course, not the best possible.

As a simple generalization of Theorem 1 we can prove also that, given a real number $K>0$, there is an $n_0=n_0(K)$ such that for any $n\geqq n_0$ and any set of n distinct positive integers $m_1, m_2,\cdots, m_n$ we have

$$\min_{0\leqq x<2\pi}\sum_{j=1}^{n}\cos(m_jx+\omega_j)<-K,$$

where $\omega_1, \omega_2,\cdots, \omega_n$ are arbitrary real numbers, and in particular,

$$\min_{0\leqq x<2\pi}\sum_{j=1}^{n}\sin m_jx<-K,\qquad \max_{0\leqq x<2\pi}\sum_{j=1}^{n}\sin m_jx>K.$$

Thus Theorem 1 is a special case of the following

Theorem 2. *Let G be a locally compact connected abelian group. Given a real number $K>0$, we can find an $n_0=n_0(K)$ such that for any $n\geqq n_0$ and any set of n distinct characters $\chi_1(x), \chi_2(x),\cdots, \chi_n(x)$ on G we have*

$$\inf_{x \text{ in } G}\operatorname{Re}\sum_{j=1}^{n}c_j\chi_j(x)<-K,$$

where $c_1, c_2,\cdots, c_n$ are arbitrary complex numbers with $|c_j|\geqq 1$ $(1\leqq j\leqq n)$.

For instance, if $\lambda_1, \lambda_2,\cdots, \lambda_n$ are arbitrary distinct positive real numbers, where $n\geqq n_0$, then we have

$$\inf_{x \text{ real}}(\cos\lambda_1x+\cos\lambda_2x+\cdots+\cos\lambda_nx)<-K.$$

2. **Some lemmas.** In order to prove the theorems we appeal to a technique by P. J. Cohen [2] developed in the investigation of a different problem, and so, to avoid ambiguity, we shall here reproduce some of his lemmas given in [2] with a slight modification.