# 112. On Paracompactness of Topological Spaces

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K. Morita [5] proved that a topological space having the weak topology with respect to a closed covering is paracompact and normal if and only if each element of the closed covering is paracompact and normal.

In this note we shall investigate the paracompactness of a topological space $X$ by the open covering of $X$.

**Theorem 1.** *Let $\{G_\alpha \mid \alpha \in \Omega\}$ be a locally finite open covering of a topological space $X$. Then $X$ is paracompact**) *if and only if each $\bar{G}_\alpha$ is paracompact.*

*Proof.* Let $\mathfrak{U}=\{U_\beta \mid \beta \in \Lambda\}$ be an arbitrary open covering of $X$. Since each $\bar{G}_\alpha$ is paracompact, for each $\alpha$ we have a locally finite open covering $\{V_\gamma \frown \bar{G}_\alpha \mid \gamma \in \Gamma\}$ of $\bar{G}_\alpha$ which is a refinement of $\{U_\beta \frown \bar{G}_\alpha \mid \beta \in \Lambda\}$ and each $V_\gamma$ is an open set of $X$. By the closedness of $\bar{G}_\alpha$, $\{V_\gamma \frown \bar{G}_\alpha \mid \gamma \in \Gamma\}$ is locally finite in $X$ and, so $\{V_\gamma \frown G_\alpha \mid \gamma \in \Gamma\}$ is a locally finite collection of open sets (in $X$). By the hypothesis of the locally finiteness of $\{G_\alpha \mid \alpha \in \Omega\}$ $\{V_\gamma \frown G_\alpha \mid \gamma \in \Gamma, \alpha \in \Omega\}$ is a locally finite open covering of $X$ which is a refinement of $\mathfrak{U}$. This completes the proof.

**Theorem 2.** *Let $\mathfrak{U}=\{G_\alpha \mid \alpha \in \Omega\}$ be a star-finite open covering of a regular $T_1$ space $X$. If each $G_\alpha$ is paracompact as a subspace of $X$ and each $\mathrm{Fr}(G_\alpha)$ has the Lindelöf property, then $X$ is paracompact.*

**Theorem 3.** *Let $\mathfrak{U}=\{G_\alpha \mid \alpha \in \Omega\}$ be a locally finite open covering of a regular $T_1$ space $X$. If each $G_\alpha$ is paracompact and each $\mathrm{Fr}(G_\alpha)$ is compact, then $X$ is paracompact.*

Proceeding the proof of the above theorems, we prove the following lemmas.

**Lemma 1.** *Let $\{G_1, G_2\}$ be an open covering of a regular $T_1$ space $X$ such that $G_i (i=1,2)$ are paracompact as subspaces of $X$. In order that $X$ is paracompact it is necessary and sufficient that there are mutually disjoint open sets $H_i (i=1,2)$ of $X$ such that $H_i \subset G_j (i \neq j; i,j=1,2)$ and $H_i \supset \mathrm{Fr}(G_i)$ $(i=1,2)$.*

*Proof.* *Necessity.* By [2] $X$ is normal. Since $\mathrm{Fr}(G_i)$ $(i=1,2)$ are closed in $X$ and are mutually disjoint, there exist open sets $H_i (i=1,2)$ satisfying the above conditions.

*) Topological space $X$ is said to be *paracompact* if each open covering of $X$ has a locally finite open covering as a refinement.