109. On the Inequality of Steiner

By Kanesiroo ISEKI

Department of Mathematics, Ochanomizu University, Tokyo

(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1960)

We shall be concerned with continuous curves in a Euclidean space $\boldsymbol{R}^m$ of any dimension m. We interpret $\boldsymbol{R}^m$ as a vector space consisting of all the m-tuples $\langle x_1,\cdots,x_m\rangle$ of real numbers, and by a *continuous curve* in $\boldsymbol{R}^m$ we understand a continuous mapping of the real line $\boldsymbol{R}$ into $\boldsymbol{R}^m$ (we may identify $\boldsymbol{R}^1$ with $\boldsymbol{R}$). In what follows, a *curve*, by itself, will always mean a continuous curve in $\boldsymbol{R}^m$ which is locally rectifiable, i.e. rectifiable on every closed interval. This will be tacitly understood throughout. It may be observed that, under this agreement, the sum of any pair of curves is likewise a curve.

We shall call *length-function* for a curve $\varphi(t)$ any real-valued function $F(t)$ defined on $\boldsymbol{R}$ and such that, for every closed interval $I=[a, b]$, the length of the curve φ over I is equal to $F(I)$, where $F(I)$ means as usual the increment $F(b)-F(a)$ of F over I. Thus $F(t)$ is continuous and monotone nondecreasing. Of course, F is not uniquely determined by φ.

Given a pair of curves φ and ψ, let F, G, H be any length-functions for φ, ψ, and $\varphi+\psi$ respectively. Then it is easy to see that, for every closed interval I, we have the relation

$$H(I)\leqq F(I)+G(I). \tag{1}$$

This is the inequality of Steiner. Now it is the object of this paper to *obtain a necessary and sufficient condition for the equality sign to hold in* (1) *for a given interval I.* Although a number of partial or intermediate results in this direction are given in Radó [1], it seems to us that no complete solution of the problem has appeared as yet.

We find it convenient to give a few more definitions. By the *direction* of any nonvanishing vector q of $\boldsymbol{R}^m$ we understand the unit-vector $|q|^{-1}q$. The latter will sometimes be denoted by the symbol $\operatorname{dir} q$. Given a curve φ and a point c of the real line, a unit-vector p of the vector space $\boldsymbol{R}^m$ will be called *tangent direction of φ at the point c*, iff (i.e. if and only if) for any positive number ε we can find another positive number δ such that, whenever I is a closed interval containing c and having length less than δ, we have both $\varphi(I)\neq 0$ and $|\operatorname{dir}\varphi(I)-p|<\varepsilon$. The tangent direction of φ at c is obviously uniquely determined when existent, and will be denoted by the symbol $\hat{\varphi}(c)$.

As a notion closely related to that of tangent direction we define further the velocity of a curve φ at a point $c\in\boldsymbol{R}$ as follows. Writing