143. *Note on H-spaces*

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1. Let G be a topological group and also a CW-complex (where the connectedness is not assumed), and p: $E\to B$ be a universal bundle with group G, where "universal" means that all the homotopy groups of E vanish. (The existence of a universal bundle is proved by Milnor [1].) Let ΩB be a loop space in B with base point $*=p(G)$ and 0_* the constant loop.

The following theorem is due to Samelson [4], essentially.

Theorem 1. *There is an H-homomorphism* f: $G\to\Omega B$, $f(e)=0_*$ (e *the unit of* G), *which is also a weak homotopy equivalence.*

Here, "H-homomorphism" means that the two maps

$$(x, y)\to f(xy) \quad \text{and} \quad (x, y)\to f(x)\circ f(y),$$

of $(G\times G, (e, e))$ into $(\Omega B, 0_*)$, are homotopic; and "weak homotopy equivalence" means that f induces isomorphisms of all the homotopy groups of G and ΩB, i.e., more precisely speaking,

(a) f_*: $\pi_0(G)\to\pi_0(\Omega B)$ is 1-1 and onto, and

(b) $(f\,|\,C(G, x))_*$: $\pi_i(C(G, x))\to\pi_i(C(\Omega B, f(x)))$ is an isomorphism for any $x\in G$ and positive integer i, where $C(X, x)$ is the arcwise-connected component of X containing $x\in X$.

Proof. Because G is a CW-complex and $\pi_i(E)=0$ for $i\geqq 0$, G is contractible in E to e, leaving e fixed. Denote such a contraction by

$$k_t\colon G\to E,\ 0\leqq t\leqq 1\colon k_0=\text{identity},\ k_1(G)=k_t(e)=e.$$

The map f: $G\to\Omega B$ is defined by

$$f(x)(t)=p\circ k_t(x), \quad \text{for } x\in G.$$

By the same proof of [4, Theorem I], it is proved that f is an H-homomorphism, noticing that $G\times G$ has the same homotopy type of a CW-complex [2, Proposition 3], and a map of a CW-complex into E is homotopic to the constant map.

Consider the diagram

$$\begin{array}{ccc} \pi_i(E, G, e) & \xrightarrow{\partial} & \pi_{i-1}(G, e) \\ \downarrow p_* & & \downarrow f_* \\ \pi_i(B, *) & \xrightarrow{T} & \pi_{i-1}(\Omega B, 0_*), \end{array}$$

where T is the natural isomorphism. p_*, T and ∂ are isomorphisms for $i\geqq 2$ and 1-1, onto for $i=1$.

For a map h: $(I^{i-1}, \dot{I}^{i-1})\to(G, e)$, define $\bar{h}$: $(I^i, \dot{I}^i, J^{i-1})\to(E, G, e)$ by $\bar{h}(s, t)=k_t\circ h(s)$ for $(s, t)\in I^{i-1}\times I$. Then