139. A Problem of Number Theory

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In this paper we shall consider a problem of number theory. In his recent book, Sto Zadan (in Polish), Prof. H. Steinhaus has solved an interesting problem on number theory: For any natural number

$$\alpha=10^{n-1}a_n+10^{n-2}a_{n-1}+\cdots+10^2a_3+10a_2+a_1$$

expressed in the decimal system, we calculate the sum of the squares of its digit of α,

$$\alpha_1=a_n^2+a_{n-1}^2+\cdots+a_3^2+a_2^2+a_1^2.$$

For the number α_1, we calculate the sum of squares of all digits contained in α_1. We repeat the same processes. If we do not reach 1, then we have a cyclic finite sequence:

145, 42, 20, 4, 16, 37, 58, 89.

This problem is generalised in the following forms: *Let k be a fixed positive integer, for any natural number*

$$\alpha=10^{n-1}a_n+10^{n-2}a_{n-1}+\cdots+10^2a_3+10a_2+a_1$$

we calculate

$$\alpha_1=a_n^k+a_{n-1}^k+\cdots+a_3^k+a_2^k+a_1^k$$

and for the integer α_1, we calculate the sum of k-powers of all digits a_i of α_1. We repeat the processes. We should like to know all *cyclic parts* appeared except the trivial case.

If such a cyclic part has the sequence with l-terms, we call it a cyclic sequence of the *length l for power k*. Then the results by H. Steinhaus are stated as follows: For $k=2$, there appear a cyclic sequence of the length 8 and a trivial sequence of the length 1.

We can prove *theoretically* that there exist finite numbers of cyclic sequences for each power k. We can not find these *individual* cyclic parts by theoretic methods, and this difficulty is therefore purely technical.

Here, we shall decide all cyclic sequences for $k=3$. The detail result will be found in the table, and its calculation was done by a small desk calculator. For $k=3$, it is seen from an easy calculation (see H. Steinhaus, loc. cit.) that we must find all cyclic sequences of numbers less than 2000, and as we can check that new cyclic sequences between 1000 and 2000 do not appear by a trivial verification, the table shows all cyclic parts from 1 to 999.