138. Weakly Compact Operators on the Spaces of Continuous Functions

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In this note we shall give a brief account of some properties of weakly compact operators on the spaces of continuous functions on general spaces. Our main purpose is to extend some results of Arens [1] and Grothendieck [5]. Full details will appear in Osaka Mathematical Journal.

1. Let E and F be locally convex linear topological spaces. Then a continuous linear operator T of E into F is said to be a *compact* (resp. *weakly compact*) *operator* if T maps a neighborhood of 0 in E into a compact (resp. weakly compact) subset in F. A completely regular Hausdorff space X is said to be a *k_0-space* if whenever $U\frown K$ is a neighborhood of x_0 in K for a subset U ($\ni x_0$) and for any compact subset K ($\ni x_0$), U is a neighborhood of x_0 in X. A neighborhood here need not be an open set. A k_0-space is a k-space,[1] and any completely regular space satisfying the 1st axiom of countability or any locally compact Hausdorff space is always ak_0-space. Let X be a topological space and $\mathfrak{S}$ be a set of compact subsets. Then we denote by $C_{\mathfrak{S}}(X)$ the space of all continuous functions on X with the topology of uniform convergence of sets in $\mathfrak{S}$. "$\bigcup\mathfrak{S}=X$" denotes that the sum of all subsets in $\mathfrak{S}$ is X.

We first extend a theorem of Bartle [2][2] to the case of locally convex topological linear spaces.

Theorem 1. (i) *Let E be a barrelled locally convex linear space. Let Y be a completely regular Hausdorff space and let $\mathfrak{S}$ be a set of compact sets in Y with $\bigcup\mathfrak{S}=Y$. Then a linear operator T of E into $C_{\mathfrak{S}}(Y)$ is continuous if and only if there is a continuous mapping τ of Y into E' with respect to the topology $\sigma(E',E)$ such that $(Te)y=\langle\tau y,e\rangle$ for any $e\in E$ and for any $y\in Y$.*

(ii) *Let E be a Banach space. Let Y be a completely regular Hausdorff space and $\mathfrak{S}$ be a set of compact subsets in Y with $\bigcup\mathfrak{S}=Y$. Then a continuous linear operator T of E into $C_{\mathfrak{S}}(Y)$ is weakly compact if and only if there is a continuous mapping τ of Y into E' with respect to the topology $\sigma(E',E'')$ such that $(Te)y=\langle\tau y,e\rangle$ for $e\in E$ and $y\in Y$.*

(iii) *Let E be a locally convex topological linear space. Let Y be*

1) Cf. for example, Kelly [6].
2) Cf. [2, p. 55, Theorem 10.2].