129. *The Diffusion Satisfying Wentzell's Boundary Condition and the Markov Process on the Boundary. I*

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1. W. Feller [1, 2] determined all the diffusion processes in one dimension. He obtained the intrinsic form of the differential operator for the diffusion equation and constructed solutions for all the possible boundary conditions. As an approach to such a solution in multi-dimensional case, A. D. Wentzell [7] formulated the problem in the following way.[1] Let D be a bounded domain in an N-dimensional orientable manifold of class C^∞ with sufficiently smooth boundary ∂D. Given a diffusion equation

$$\frac{\partial}{\partial t}u = Au, \tag{1}$$

$$\begin{aligned} Au(x)=a(x)^{-\frac{1}{2}}\sum_{i,j=1}^{N}\frac{\partial}{\partial x^i}\Big(a^{ij}(x)\,a(x)^{\frac{1}{2}}\frac{\partial u(x)}{\partial x^j}\Big) \\ -a(x)^{-\frac{1}{2}}\sum_{i,j=1}^{N}\frac{\partial}{\partial x^i}(b^i(x)\,a(x)^{\frac{1}{2}}u(x))+c(x)u(x), \end{aligned} \tag{2}$$

where $\{a^{ij}(x)\}$ and $\{b^i(x)\}$ are contravariant tensors on $\bar{D}$ of class C^3, $\{a^{ij}(x)\}$ is symmetric and positive definite for each $x\in\bar{D}$, $c(x)\in C^2(\bar{D})$ is non-positive and $a(x)=\det(a_{ij}(x))$ where $a_{ij}(x)$ is the conjugate covariant tensor of $(a^{ij}(x))$. $C^n(\bar{D})$ and $C^n(\partial D)$ are the spaces of n-times continuously differentiable functions on $\bar{D}$ and ∂D respectively. We write $C(\bar{D})$ and $C(\partial D)$ for $C^0(\bar{D})$ and $C^0(\partial D)$ respectively. $C^H(\bar{D})$ is the space of uniformly Hölder continuous functions on $\bar{D}$. The norms of these spaces are those of uniform convergence. Operator A, defined on $C^2(\bar{D})$, has the closure $\bar{A}$ in $C(\bar{D})$. Now, the problem is to *find all the semi-groups* $\{T_t, t\geq 0\}$ *of non-negative linear operators* T_t *on* $C(\bar{D})$ *with norm* $\|T_t\|\leq 1$, *strongly continuous in* $t\geq 0$, *and having a contraction of* $\bar{A}$ *as its generator* $\mathfrak{G}$ *in Hille-Yosida sense.* Wentzell proved that, for any $u\in C^2(\bar{D})\frown\mathfrak{D}(\mathfrak{G})$ and $x_0\in\partial D$ we have

$$Lu(x_0)=0,\quad x_0\in\partial D, \tag{3}$$

$$\begin{aligned} Lu(x)=\sum_{i,j=1}^{N-1}\alpha^{ij}(x)\frac{\partial^2 u(x)}{\partial\xi_x^i\xi_x^j}+\sum_{i=1}^{N-1}\beta^i(x)\frac{\partial u(x)}{\partial\xi_x^i}+\gamma(x)u(x) \\ +\delta(x)\lim_{y\to x}Au(y)+\mu(x)\frac{\partial u(x)}{\partial n}+\int_{\bar{D}}\Big\{u(y)-u(x)-\sum_{i=1}^{N-1}\frac{\partial u(x)}{\partial\xi_x^i}\xi_x^i(y)\Big\}\nu_x(dy), \end{aligned} \tag{4}$$

1) The following set up is slightly modified for our present use.