149. On Relative Derivation of Additive Set-Functions

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1. **Introduction.** This is a continuation of the paper [1] by the same author. Utilizing the extension of the Vitali covering theorem established there, we shall obtain in what follows a series of consequent theorems concerning relative derivation of additive set-functions. The greater part of our results will, however, be analogues of the corresponding results contained in the fourth chapter of Saks [2] (we shall henceforth quote this book simply as Saks for short). We shall omit the proofs whenever they are mere repetitions, at most with slight modifications, of those given in Saks. We remark explicitly that, as in [1], the space that underlies all our considerations is the real line $\boldsymbol{R}$.

2. **Preliminary remarks.** We shall be interested in finite real set-functions which are defined and completely additive on the class of the bounded Borel subsets of the real line $\boldsymbol{R}$. For brevity they will simply be called *additive set-functions.*

Suppose we are given a nonnegative additive set-function μ. We shall keep this notation fixed throughout the rest of the paper. As is well known (*vide* p. 71 of Saks), we can attach to μ a finite nonnegative additive interval-function F, defined for linear closed intervals and such that $F^*(X)=\mu(X)$ for all bounded Borel sets X in $\boldsymbol{R}$, where F^* means the outer measure of Carathéodory induced by F.

Now simple examples show that in general the interval-function F is not uniquely determined by μ. However, the outer measure F^* itself is uniquely determined by μ. To see this we observe firstly that for any set E (bounded or not) the value of $F^*(E)$ is equal to the infimum of $F^*(D)$ for open sets $D\supset E$, as stated on p. 68 of Saks. But here $F^*(D)$ must be independent of the choice of F, since F^* and μ coincide for bounded Borel sets and since D can be expressed as the limit of an ascending sequence of bounded open sets.

We are thus entitled to write henceforth $\tilde{\mu}$ for the function F^*, and all the results given in Saks on outer measures of Carathéodory and on measures induced by nonnegative additive interval-functions, as well as our extension of the Vitali theorem, will hold good for $\tilde{\mu}$. Moreover, it follows from what has been said above that we may also construct the function $\tilde{\mu}$ as follows: the value $\tilde{\mu}(D)$ for an open set D is by definition the supremum of the values of μ for bounded open sets