6. A Certain Type of Vector Field. I

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(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1961)

I. Let M be a differentiable manifold of class C^3 and of dimension n and assume that M has an affine connection. We denote covariant differentiation by $X \in T_x$ with respect to this affine connection by ∇_X provided that T_x is the tangent vector space at $x \in M$. Then to each vector field V defined in a neighborhood of x there is attached a homomorphism A_V of T_x into itself, provided that for $X \in T_x$, $A_V(X)$ is defined as usual to be $-\nabla_X V$.

It is expected that there will exist certain correspondences between the geometric natures of vector field V and the algebraic natures of A_V. The objective of the present paper is to study these correspondences in a certain case which will be stated below and to arrange the preliminaries of the author's previous paper [1][1] which has determined both the metrics and the topological types of the complete manifolds admitting a torse-forming vector field with some singularities. It is interesting to note that these manifolds show remarkable similarities, both in their metric aspects and in their topological aspects, to the hypersurfaces of rotation which admit at least one torse-forming vector field as is shown in the sequel.

Let f be a differentiable function defined in a neighborhood of x and consider the germ defined by f which will be denoted by $[f]$. The total of $[f]$ forms a ring, denoted by $\mathfrak{S}$. Let $\mathfrak{S}[\eta]$ denote a polynomial ring over coefficient ring $\mathfrak{S}$. Further let Grad be a linear map of module $\mathfrak{S}$ into T_x^* which to each $[f]$ assigns the value at x of the gradient covector of f.

If M has a Riemann metric $g(X, Y)$ $(X, Y \in T_x)$, we can define a linear map $P_Z (Z \in T_x)$ by $P_Z(X) = g(X, Z)/g(Z, Z)Z$. Then one of the simplest types of A_V is the one with $A_V \in \mathfrak{S}[P_V]$.[2] If we write

$$\mathfrak{S}^* = \{[f] \mid P_V \circ \operatorname{Grad}[f] = \operatorname{Grad}[f]\},$$

then $\mathfrak{S}^*$ is a subring of $\mathfrak{S}$ and if R denotes the real number field, then R also is a subring of $\mathfrak{S}$ in the natural sense. Let $\mathfrak{P}$ denote a specialization: $\mathfrak{S}[\eta] \ni s(\eta) \to s(0) \in \mathfrak{S}$. Then $\mathfrak{P}^{-1}(\mathfrak{S}^*)$ is a subring of $\mathfrak{S}[\eta]$ or of $\mathfrak{S}[P_V]$ when η is regarded as P_V and, similarly, $\mathfrak{P}^{-1}(R)$ is a subring of them.

The vector fields discussed in the present paper are the gradient

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1) Numbers in brackets refer to the reference at the end of the paper.

2) More precisely, $A_v = i_x(f(P_v))$ for some $f \in \mathfrak{S}[\eta]$, where i_x means a map assigning the value at x of s to each $[s]$.