4. On Poisson Integrals

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1. Let $f(t)$ be an integrable function on the interval $[-\pi, \pi]$, then we can consider the Poisson integral

$$(1)\qquad u(re^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1-r^2}{1+r^2-2r\cos(t-\theta)}dt \quad (0\le r<1,\ 0\le\theta<2\pi).$$

The following theorem concerning the Poisson integral is well known: *if $f(t)$ has a derivative at $t=\theta_0$, then we have* $\lim\limits_{r\to1}\frac{\partial u(re^{i\theta_0})}{\partial\theta}=f'(\theta_0)$. The purpose of this paper is to investigate whether this theorem holds for other derivatives. As

$$(2)\qquad \frac{\partial u(re^{i\theta})}{\partial\theta}=\frac{-1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\partial}{\partial t}\Big(\frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\Big)dt,$$

we shall consider the integrals of this type.

2. We shall begin with the positive result.

THEOREM 1. *If $f(t)$ has a symmetric Borel derivative*[1] *at θ_0, then we have* $\lim\limits_{r\to1}\frac{\partial}{\partial\theta}u(re^{i\theta_0})=B_s'f(\theta_0)$.

Proof. Without loss of generality, we can assume that $\theta_0=0$ and $B_s'f(\theta_0)=0$. If we set $F(t)=\int_t^{\pi}\frac{f(t)-f(-t)}{2t}dt$, $F(h)=F(0)+h\varepsilon(h)$, it follows from the hypothesis that for every $\varepsilon>0$ there exists δ such that $0\le h<\delta$ implies $|\varepsilon(h)|<\varepsilon$. Fixing δ we divide the integral (2) into three parts:

$$\frac{-1}{2\pi}\Big[\int_{-\pi}^{-\delta}+\int_{-\delta}^{\delta}+\int_{\delta}^{\pi}\Big]=\frac{-1}{2\pi}(I_1+I_2+I_3).$$

Integration by parts leads to the evaluation of I_3,

$$|I_3|\le M\cdot\frac{1-r}{4r\sin^4\delta/2}+M\int_{\delta}^{\pi}\Big|\frac{\partial^2}{\partial t^2}\Big(\frac{1-r^2}{1+r^2-2r\cos t}\Big)\Big|dt\le K(1-r),$$

where $M=\int_{-\pi}^{\pi}|f(t)|dt$, K is a constant not depending on r. Therefore

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1) A function $f(t)$ has a Borel derivative $\alpha(\neq\infty)$ at θ_0 if $\lim\limits_{h\to0}\frac{1}{h}\int_0^h\frac{f(t+\theta_0)-f(\theta_0)}{t}dt=\alpha$ and we write it $B'f(\theta_0)$. Similarly $f(t)$ has a symmetric Borel derivative $B_s'f(\theta_0)=\alpha$ at θ_0 if $\lim\limits_{h\to0}\frac{1}{h}\int_0^h\frac{f(\theta_0+t)-f(\theta_0-t)}{2t}dt=\alpha$, where the integrals are taken in the sense of $\lim\limits_{\varepsilon\to0}\int_{\varepsilon}^{h}$.