2. On Convergences of Sequences of Continuous Functions

By Takesi ISIWATA
Tokyo Gakugei University

(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1961)

We assume here that X is always a completely regular T_1-space and functions are all continuous and real-valued. It is well known that various types of convergences are defined for sequences of continuous functions. In this paper, we shall give the characterizations of spaces in which one of eight types of convergences, listed in § 1, implies some other types of convergences.

1. **Definitions.** We denote by $\{f_n\}\to f$ that $\{f_n\}$ converges to f *pointwisely* on X. We denote $\{f_n\}\to f$ (*unif.*) that $\{f_n\}$ converges *uniformly* to f. $\{f_n\}\to f$ (*compact*) means that $\{f_n\}$ is uniformly convergent to f on every compact subset of X. We shall say that $\{f_n\}$ *converges locally uniformly* to f ($\{f_n\}\to f$ (*loc. unif.*)) if for every $\varepsilon>0$ and every point x, there is an open neighborhood U of x and some integer $m>0$ such that $U\ni y$ implies that $|f_n(y)-f(y)|<\varepsilon$ for every $n>m$. $\{f_n\}$ is said to be *strictly continuously convergent* to f ($\{f_n\}\to f$ (*str. cont.*)) if $\{f(x_n)\}\to\alpha$ implies $\{f_n(x_n)\}\to\alpha$. $\{f_n\}$ is said to be *continuously convergent* to f ($\{f_n\}\to f$ (*cont.*)) if $\{x_n\}\to x$ implies $\{f_n(x_n)\}\to f(x)$. f is said to be *convergent to f quasi-uniformly* ($\{f_n\}\to f$ (*quasi-unif.*)) if $\{f_n\}\to f$ and if for every $\varepsilon>0$ and $m>0$, there exists a finite number of n_i, $n_i>m$ $(i=1,2,\cdots,k)$ such that for every x in X, $|f_{n_i}(x)-f(x)|<\varepsilon$ holds for at least one n_i. $\{f_n\}$ is said to be *almost uniformly convergent* to f ($\{f_n\}\to f$ (*almost unif.*)) if every subsequence of $\{f_n\}$ is quasi-uniformly convergent to f.

In the following, in case one type of convergences implies always the other one type of convergences, for instance, $\{f_n\}\to f$ (*unif.*) implies always $\{f_n\}\to f$, then we write [*unif.*]$\to$[*pointwise*]. If U is an open set, then a non-negative continuous function f such that $f=0$ on $X-U$, $0\le f\le 1$ and $f(x)=1$ for some x in U is said to be an *associated function with U*. For a given U containing two points at least, there are many associated functions. If f is an associated function with an open set U, then $V=\{x;f(x)>1/2\}$ is said to be an *f-section* of U. Let $\{U_n\}$ be a family of open sets and let f_n be an associated function with U_n; then $\{f_n\}$ is said to be a *sequence associated with* $\{U_n\}$.

2. **Constructions of sequences of functions.** (2.1) *Suppose that X is not finite.* Then there is a sequence $\{x_n\}$ such that each point x_n is isolated in $\{x_n\}$. Thus there is, by the regularity, a family $\{U_n\}$