25. *On the Unitary Equivalence of Normal Operators in Hilbert Spaces*

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The purpose of this paper is to find a necessary and sufficient condition for the unitary equivalence of normal operators in the abstract Hilbert space $\mathfrak{H}$ which is complete, separable, and infinite-dimensional.

Definition. If we denote by $\mathfrak{M}^{\alpha}$ the eigenspace determined by all eigenelements of a normal operator N in $\mathfrak{H}$ corresponding to the eigenvalue α, the projection operator of $\mathfrak{H}$ on $\mathfrak{M}^{\alpha}$ is called the eigenprojector corresponding to the eigenvalue α of N.

Theorem 1. Let N_1 and N_2 be normal operators in $\mathfrak{H}$ such that the sum of all eigenprojectors of N_j is identical with the identity operator I for each value of $j=1, 2$. Then for the unitary equivalence of N_1 and N_2 it is necessary and sufficient that N_1 and N_2 have the same continuous spectrum and same point spectrum (inclusive of the multiplicities of eigenvalues).

Proof. From the fact that the spectral classification of the points on the complex plane for N_j (inclusive of the multiplicities of eigenvalues) is invariant under the unitary transformation UN_jU^{-1} for any unitary operator U, it is clear that the condition given in the theorem is necessary; hence it remains only to prove the sufficiency of the condition.

Let $\{\varphi_\nu^{(j)}\}$ be an orthonormal set of all eigenelements of N_j; let $\{l_n\}$ and Δ be the common point spectrum and common continuous spectrum of N_1 and N_2 respectively; and let $\{P_j(z)\}$, $\{E_j(\lambda)\}$ and $\{F_j(\mu)\}$ be the spectral families of N_j, the self-adjoint operators $H_j=\frac{1}{2}(N_j+N_j^*)$ and $K_j=\frac{1}{2i}(N_j-N_j^*)$ respectively. Then, by hypotheses, $\{\varphi_\nu^{(1)}\}$ and $\{\varphi_\nu^{(2)}\}$ are complete orthonormal sets respectively and can be put in one-to-one correspondence in such a way that corresponding elements are eigenelements for N_1 and N_2 respectively, corresponding to the same eigenvalue; and in addition, since the residual spectrum of N_j is empty and since the spectral representation of N_j vanishes on the resolvent set,

$$N_j=\sum_n l_nP_j^{(n)}+\int_\Delta zdP_j(z), \quad H_j=\sum_n \Re(l_n)P_j^{(n)}+\int_\Delta \Re(z)dP_j(z),$$