19. Note on a Semigroup Having No Proper Subsemigroup

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In the previous paper [1] we determined the structure of $\mathfrak{S}$-semigroup and added that a finite semigroup of order>2 which contains no proper subsemigroup is a cyclic group of prime order. Further we noticed, without proof, that this holds even if the condition "finiteness" is excluded.

In the present note we shall prove the theorem without using the result of $\mathfrak{S}$-semigroup.

Theorem. *A semigroup of order*[1]>2 *which has no proper subsemigroup is a cyclic group of prime order.*

Let S be a semigroup of order>2 which has no proper subsemigroup, and let a and b be arbitrary distinct elements of S. Then we see that S is generated by a and b. First we must prove that S contains at least a non-idempotent element. For this purpose we may show that an idempotent semigroup M of order>2 generated by the two distinct elements a and b has at least one proper subsemigroup. Now, F denotes the free idempotent semigroup generated by a and b. M is given as a suitable[2] factor semigroup of F. Fortunately it is easily proved[3] that F is a semigroup of order 6 which consists of

$$a, \quad b, \quad ab, \quad ba, \quad aba, \quad bab.$$

As is easily seen, F has a proper subsemigroup, say $\{b, ab, ba, aba, bab\}$. Let us consider every decomposition[4] of F which raises a factor semigroup M.

If a or b alone composes a coset, the problem is clear, that is, if $\{a\}$ is a coset, all the other cosets form a proper subsemigroup of the factor semigroup M, because $\{b, ab, ba, aba, bab\}$ is a subsemigroup of F. If a and b belong to different cosets containing at least two elements, then we may examine only the decomposition such that

$$(1) \qquad a\sim aba, \quad b\sim bab, \quad \text{and} \quad a\not\sim b,$$

because the other factor semigroups of F would be of order at most 2. In detail,

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1) By "a semigroup of order>2" we mean "an infinite or finite semigroup which contains at least 3 elements".

2) We require a condition that a and b do not belong to the same coset.

3) See [2].

4) A classification of elements which gives a factor semigroup is called a decomposition of a semigroup. Each decomposition corresponds to each congruence relation.