18. On (m, n)-distributive Division Rings

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Following Beaumont [1], an additive abelian group R, closed with respect to multiplication, is called an *(m, n)-distributive ring* if the (m, n)-distributive law holds in R:

(1) $$(\sum_{i=1}^{m} a_i)(\sum_{j=1}^{n} b_j)=\sum_{i,j=1}^{m,n} a_i b_j \quad \textit{for all } a_i, b_j \in R.$$

In other words, an (m, n)-distributive ring is a system arising from the definition of a (not necessarily associative) ring by replacing two distributive laws by the above (m, n)-distributive law. The structure of (m, n)-distributive rings was studied by Beaumont [1], Hsiang [2] and Saitô [3].

An (m, n)-distributive ring D is called an *(m, n)-distributive division ring* if D has at least two elements and $D-\{0\}$ forms a multiplicative group. In this note, we study the connection between (m, n)-distributive division rings and (ordinary) division rings.

We consider exclusively (m, n)-distributive rings for $m, n \geqq 2$. So, in this paper, *we assume that m and n are always integers $\geqq 2$.*

Theorem 1. *If an (m, n)-distributive division ring D contains at least three elements, then D is a division ring.*

Proof. Recall, in an (m, n)-distributive ring, we have

(2) $$(a+b)c=ac+bc-0c, \quad c(a+b)=ca+cb-c0$$

([1], p. 877). By assumption, for any $x \in D^*=D-\{0\}$, we can take an element $y \in D^*$ such that $y \neq x$. We set $z=yx^{-1}$. Then $z-1 \in D^*$. Now, by the equality (2) and the associativity of multiplication in D^*, we have

$$\begin{aligned} yx^{-1}+(0x)x^{-1}-0x^{-1} &= (y+0x)x^{-1}=(zx+0x)x^{-1} \\ &= (((z-1)+1)x+0x)x^{-1} \\ &= ((z-1)x+1x-0x+0x)x^{-1} \\ &= ((z-1)x+x)x^{-1} \\ &= (z-1)xx^{-1}+xx^{-1}-0x^{-1} \\ &= (z-1)+1-0x^{-1}=yx^{-1}-0x^{-1}. \end{aligned}$$

Hence $(0x)x^{-1}=0$. By the closedness of multiplication in D^*, we have $0x=0$. Dually $x0=0$. Moreover, since

$$0=0z=0((z-1)+1)=0(z-1)+01-00=0-00,$$

we have $00=0$. Thus

$$0x=x0=0 \quad \text{for all } x \in D.$$

Hence, by (2), we have distributive laws:

$$(a+b)c=ac+bc, \quad c(a+b)=ca+cb.$$

This completes the proof.